THE

ELEMENTS

OF

LINEAR PERSPECTIVE,

DESIGNED

FOR THE USE OF STUDENTS

IN THE

UNIVERSITY.

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PREFACE.

THE following pages originated in the supposed want of a concise treatise on Perspective, adapted to the system of education established in the University of Cambridge. Perhaps no subject, within the whole range of mathematical enquiry, is in itself more attractive. That some degree of skill in drawing might be acquired by repeated trials, and long practice; that the study of pictures might furnish some helps toward copying original objects, is easily understood: But it is not so readily conceived how the business of delineation can be reduced to a science, certain and demonstrable as that of arithmetic. The principles, by means of which this is effected, although few, and plain, and familiar to the studious, lie beyond the limits of common observation; neither is such an application of them very likely to occur to those who know them best. The bare enunciation, therefore, of the problem, *so to represent an object upon a given surface, that the* *picture and its original shall excite the same sensations*, is sufficient to stimulate the curiosity of a young and ingenuous mind. Whether, indeed, the doctrine of Perspective be considered only as a remarkable instance of ingenious speculation, or as forming the basis of correct design, and instructing the judgment of the connoisseur in painting, it comes sufficiently recommended to the man of liberal education.

A request has, in fact, been more than once made to the author of this essay, by those whom he assisted in their academical studies, to point out some work on this subject for their perusal. With that request he found it difficult to comply. For the larger works on Perspective, although some of them be very valuable, contain much more than is wanted for common use; and it is troublesome to select from them what ought to be read: Whilst the smaller treatises, with the exception of the second edition of that of Dr. Brook Taylor, published in 1719, are, generally speaking, deficient in methodic arrangement and accuracy of demonstration. In most of them, the definitions are loosely expressed, and laid down in greater numbers than the case absolutely requires; for the geometrical truths which are cited the reader is seldom referred to the Elements of Euclid, the standard book of reference in this University; the propositions are often stated, not in general terms, but, to accommodate the indolence of the writer, as relating to a particular diagram: and, what is of more consesequence, principles which are really general are often not shewn in their full extent, but as belonging to the most usual situation of the Perspective Plane: The learner is perplexed by too great a variety of methods, instead of having his attention directed to those which may be deemed the most perspicuous, and the best in practice: What is called the *Geometrical Plane* is commonly employed, notwithstanding that the use of it, in most cases, increases the number lines to be drawn, and is never resorted to but at the expence of neatness and simplicity: And, lastly, the inverse problems of Perspective are most frequently omitted, although highly interesting in themselves and admitting of the most elegant constructions. These observations, let it be remarked, apply to the labours of those who have treated the subject *mathematically:* No notice is intended to be taken of the popular works which, from time to time, are published, which give rules without investigations, and profess to perform what is impossible, to put their readers in possession of the principles of Perspective, without requiring from them the previous knowledge of geometry.

The considerations, which seemed to justify the following attempt, have thus been fully explained. But, even if the faults above specified shall have been avoided, the plan which has been adopted may, perhaps, be thought too narrow. It is obvious, however, to remark, that much of the time of a student, unless painting be his profession, ought not to be spent upon so slender a branch of learning: Yet, it undoubtedly exhibits so luminous and useful an application of geometry, as to claim the attention of every well educated person. Its principles are universally worth knowing; its difficulties and minuter details are more properly the study of the professed artist, than of the general scholar. The first six books, and part of the eleventh, of Euclid's Elements, are the only books necessary to be read, in order that the whole, of what is here delivered, may be understood. Thus much of Perspective, it is presumed may furnish at least an useful exercise; affording a more easy and engaging application of rudiments previously acquired, than many other subjects supply; and more than is here taught can seldom be wanted in practice. The operations of

Perspective are, in reality, too complicated to make it advantageous to apply them strictly to the delineation of any objects but such as are terminated by straight lines. They who wish to copy accurately and expeditiously the scenery of nature, will probably have recourse to mechanical means; and the Camera Lucida, the recent invention of Dr. Wollaston, will be found well suited to their purpose. Still, whoever employs himself in drawing will find his advantage in learning the principles of Perspective. There is one class of literary characters, the authors of books of Travels made expensive by their engravings, of whom the public have a right to expect this attainment. To them, indeed, the want of it is more disgraceful, than the possession of it would have been creditable. For it has been justly remarked, that there is the same absurdity in pretending to delineate, without a knowledge of Perspective, as in attempting to write an epic poem, without being acquainted with the rules of orthography and grammar. To some of the latest and most expensive publications of Travels may be objected the inaccuracy of the drawing in their plates; a censure which a very small portion of information might have prevented.

Upon deliberation, no extension of the plan here adopted appeared advisable, nor did even an addition to the small number of examples,* which are given, seem to be necessary. A section on what is called *Military Perspective*, or more properly the Orthographic Projection, may perhaps be thought to have been injudiciously omitted. This, however, considered as belonging to Perspective, is a case altogether imaginary. Its laws may, indeed, be very readily investigated, by the help of what is demonstrated in the following pages, if the object be supposed indefinitely small, and the distance of the Point of View finite. This method is employed by Lambert in an Essay on Perspective, remarkable for masterly illustration and ingenious modes of operating. But in thus deducing the laws of the Orthographic Projection, the notion of INFINITE must necessarily be introduced; which notion it seemed best to exclude from a work, where precision is a main object; and which is intended for those who apply themselves to the mathematics, not less for the sake of disciplining the mind, than of storing it with knowledge.

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* *Un* exemple suffit pour donner la raison de chaque operation, de quelque espece qu 'elle soit : si vous avez besoin de plusieurs, ce n'est pas pour apprendre a opèrer: C'est seulement pour opèrer avec plus de facilitè et de promptitude ; et, avec quelque lenteur que vous procèdiêz, vous savez faire, si vous savez ce que vous faites. Exercez vous, donc, sans maitre. Ne le pouvez vous pas? Restez dans l'ignorance: C'est un oreiller assez doux pour bien des têtes. Condillac.

If the investigations here offered to the University should appear difficult to any student of the second year, that is, if they should require to be considered with a degree of attention which is fatiguing to him, it can only be attributed to one, or both, of these two causes; the newness of the terms employed, and the want of a familiar acquaintance with the first principles of the geometry of solids. It is needless to say how these causes of difficulty may be removed. It could not, however, be too strongly recommended, to those who are desirous of becoming expert in the operations of Linear Perspective, to apply the rules, as they acquire them, to easy cases of their own chusing; and to draw the figures for themselves. Various instruments, and more particularly the proportional compasses, have been contrived for facilitating the constructions which this practice requires: The use of these instruments may be learnt from books which profess to describe them. But whether expertness in the practical part of Perspective be thought worth the trouble of acquiring or not, if its Principles be well understood, the taste will be formed, and the judgment directed in drawing, even when its rules are not followed with minute precision.

It only remains formally to acknowledge, what would doubtless be inferred from the history of

Perspective, that the following is chiefly to be considered as a new work in what regards its language, the formation and connexion of its propositions, and its general arrangement. They who have read the admirable essay of Dr. Brook Taylor, will no more expect any thing which really deserves the name of originality here, than they would in a treatise on Optics, written after that of Sir Isaac Newton: And this is a subject the utmost limits of which are discovered at a first view. So that little else is left, for those who follow the above mentioned eminent mathematician, to aim at, but the merit of being more perspicuous and more extensively useful. Ego vero, neque, alienis indicibus mutatis, interposito nomine meo id profero corpus; neque ullius cogitata vituperans, institui ex eo me approbare. Sed omnibus scriptoribus infinitas ago gratias, quod, egregiis ingeniorum solertiis ex ævo collocatis, abundantes aliis alio genere copias præparaverunt. Unde nos, uti fontibus haurientes aquam, & ad propria proposita traducentes, fæcundiores & expeditiores habemus ad scribendum facultates: talibusque confidentes autoribus, audemus institutiones novas comparare.

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ELEMENTS

OF

LINEAR PERSPECTIVE.

DEFINITIONS.

I.

1. LINEAR PERSPECTIVE is that branch of Geometry in which is taught, upon a given surface, so to represent the visible outline of any given object, that the representation and its original when seen from the same point, may produce the same effect upon the organ of vision.

II.

2. In the following elementary tract, the surface, on which objects are required to be delineated, is supposed to be plane, and to be situated between the objects and the eye which views them: and it is called the Perspective Plane.

III.

3. The Point of View is that point from which the objects are seen, when they are delineated upon the Perspective Plane.

IV.

4. The point in which the Perspective Plane is met by a straight line drawn perpendicular to it from the Point of View, is called the Center of the Picture, or the Principal Point.

V.

5. The distance between the Point of View and the Principal Point, is called the Distance of the Picture, or the Principal Distance.

VI.

6. The plane which contains any original point or line, i. e. any point or line the representation of which is required, is called an Original Plane.

VII.

7. The Base Line of any Original Plane, is the common intersection of that plane, and the Perspective Plane.

VIII.

8. The point in which the Perspective Plane is met by a straight line drawn through the Point of View parallel to any original straight line, is called the Vanishing Point of that original line.

IX.

9. The intersection of the Perspective Plane, and a plane passing through the Point of View parallel to any Original Plane, is called the Vanishing Line of that Original Plane.

10. *Cor.*—The Vanishing and the Base Lines

of any Original Plane, are parallel to each other. (Art. 7. and E.* 16. 11.)

X.

11. The point in which any Vanishing Line is met by a straight line drawn perpendicular to it from the Point of View, is called the Center of that Vanishing Line.

XI.

12. The distance between the Point of View and the Center of any Vanishing Line, is called the Direct Distance of that Vanishing Line.

The definitions will be better understood by a reference to fig. 1, where *O* is the Point of View, *PL* the Perspective Plane, *XY* an Original Plane, and *Md* its parallel; *C* the Center of the Picture, and *OC* the Principal Distance; *HL* the Base Line of *XY*, and *Dd* its Vanishing Line; *K* the Center of *Dd*, and *OK* its Direct Distance.

Postulate.

Let it be supposed that the particles of light, by which objects are visible, proceed in straight lines from the objects to the eye.

It is by this Postulate that Perspective is connected with Geometry. The position which it contains is sufficiently established by unexceptionable experiment, and would be *mathematically* true, if the atmosphere were of uniform density.

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* The letter E denotes the Elements of Euclid; by the first figure or figures following it, the Proposition is denoted, and by the last the Book: The edition referred to is that of Simson.

SECTION I.

On the Delineation of Given Objects upon a Given Plane Surface.

PROP. I.

13. THEOREM.—If a straight line be drawn from the Point of View to any given visible point of an Original Object, its intersection with the Perspective Plane is the representation of that given point.

Let O (fig. 2.) be the Point of View, PL the Perspective Plane, and A any given point in an original object; join AO, and let AO meet PL, in a; a is the representation of the given point A.

For, by the postulate, the point A is visible to an eye at O, by means of particles of light proceeding from A to O, in a straight line: it is manifest, therefore, that whether the ray, by which the given point is visible, proceed from A, or a, to the eye at O, the effect upon that organ is the same; therefore, (art. 1.) a is the representation of A.

14. *Cor.* 1. If any straight line, being produced, pass through the Point of View, its representation

is a point; and that of any other straight line is a straight line.

The former part of the corollary is manifest, from the postulate: but let *AB* be any other straight line, which does not, when produced, pass through *O*; join *BO*: the collection of rays by which *AB* is visible, form (E. 2. 11.) the plane *OAB*; and *ab*, the intersection of this plane and the Perspective Plane, is its representation: But (E. 3. 11.) the common section of two planes is a straight line.

15. *Cor.* 2. If $\alpha\beta$ be the intersection of the plane *OAB* continued, and a plane *pl*, which is parallel to *PL*, and on the contrary side of *AB*, $\alpha\beta$ is also the representation of *AB* to an eye at *O*. And since the two straight lines *ab* and $\alpha\beta$ are in the same plane $O\alpha\beta$, and do not meet, although produced ever so far both ways, (E. Def. 8. 11.) the planes *PL* and *pl* being parallel to each other, therefore $\alpha\beta$ is parallel to *ab*, and (E. 29. 1. and 4. 6.) the two triangles *Oab* and $O\alpha\beta$ are similar.

$$\therefore\ O\alpha : \alpha\beta :: Oa : ab$$
$$\&\ O\alpha : Oa :: \alpha\beta : ab \text{ (E. 16. 5.)}$$

But $O\alpha$ is greater than *Oa*; therefore $\alpha\beta$ is greater than *ab*, (E. 14. 5.) and may exceed both *ab* and *AB* in any given ratio; i. e. the picture, in this case, may exceed the original line, in any given ratio, and still be a true representation of it.

PROP. II.

16. *Theorem.* The representation of a given finite straight line* parallel to the Perspective

---

* A straight line is parallel to a plane, when, both being produced ever so far both ways, the line does not meet the plane.

Plane, is parallel to that given line: And the length of the original *is to* the length of the representation, *as* the distance of the Point of View from either extremity of the original, *is to* its distance from the representation of that extremity.

Let $\alpha\beta$ (fig. 2.) be the given original finite straight line, and *ab* the intersection of the triangle $O\alpha\beta$, and the Perspective Plane *PL:* Therefore (art. 14.) *ab* is the representation of $\alpha\beta$, and it is parallel (note, and E. Def. 35. 1.) to $\alpha\beta$. Again, it may be shewn, as in art. 15, that $O\alpha : Oa :: \alpha\beta : ab$, *a* being the representation of α. In the same manner it may be proved that $O\beta : Ob :: \alpha\beta : ab$, *b* being the representation of β.

17. *Cor.* 1. If an original straight line be parallel to any straight line in the Perspective Plane, its representation is also parallel (E. 9. 11.) to that line; and if the representation of an original straight line, lying in a given plane, be parallel to the Base Line of that plane, the original is also parallel to the Base Line.

18. *Cor.* 2. If any number of parallel straight lines be all parallel to the Perspective Plane, their representations are parallel to each other.

For the representation of any one of them is (art. 16.) parallel to its original, and therefore (E. 9. 11.) parallel to all the rest, and also to their representations.

19. *Cor.* 3. The length of any straight line situated in a plane parallel to the Perspective Plane, *is to* the length of its representation, *as* the perpendicular distance of the Point of View from the plane containing the given line, *is to* the Principal Distance.

Let the straight line $\alpha\beta$ (fig. 2.) be in a plane pl, which is parallel to the Perspective Plane PL, and let ab be its representation; also, let Od, drawn perpendicularly (E. 11. 11.) to PL, and which is, therefore, (art. 5.) the Principal Distance, meet, when produced, pl in δ: Then $O\delta$ is perpendicular to pl; and $\alpha\beta : ab :: O\delta : Od$.

For since pl is parallel to PL, it follows from the demonstration of E. 15. 11, that $O\delta$ is perpendicular to pl: Join ad, $\alpha\delta$; the angles at d and δ are (E. Def. 1. 11.) right angles; and the angle $\alpha O\delta$, is common to the two triangles Oad, $O\alpha\delta$, therefore (E. 32. 1. and 4. 6.) the triangles are similar;

& $O\alpha : O\delta :: Oa : Od$
& $O\alpha : Oa :: O\delta : Od$ (E. 16. 5.)
But (art. 16.) $O\alpha : Oa :: \alpha\beta : ab$.
& $\alpha\beta : ab :: O\delta : Od$ (E. 11. 5.)

20. *Cor.* 4. Hence, the representations of all equal straight lines in a plane parallel to the Perspective Plane, are equal to one another.

For the representations will have (art. 19.) each the same ratio to its original; and since, by the supposition, the originals are equal, the representations (E. 9. 5.) will be equal also.

21. *Cor.* 5. The representation of a plane rectilineal figure parallel to the Perspective Plane, is a similar figure, and has to its original the duplicate ratio of that which the Principal Distance has to the perpendicular distance of the Point of View from the Original Plane.

For the angles of the representation are respectively equal to the corresponding angles of the original, (E. 10. 11.) because the sides containing the one, are respectively parallel (art. 16.) to the corresponding sides which contain the other.

And the sides, about equal angles in each, are proportional; because any one side *is to* its representation, (art. 19.) *as* the side adjacent *is to* its representation; and therefore (E. 16. 5.) the representations are to each other as their originals; and it was shewn that the angles contained by the former, are equal to the angles contained by the latter; the two figures are therefore (E. Def. 1. 6.) similar, and (E. 20. 6.) are to each other in the duplicate ratio of their homologous sides, that is (art. 19.) in the duplicate ratio of the perpendicular distances of the Point of View from the Perspective and the Original Planes.

22. *Remark.* It is manifest from art. 13, that the doctrine of Linear Perspective might be considered as belonging to that of the section of planes and solids. Thus the geometry of the conic sections might be readily applied to some cases of Perspective; and what is demonstrated in art. 21, is contained in this general proposition: "*If any cone or pyramid be cut by two parallel planes, the sections shall be similar figures.*" And if the cone become a cylinder, and the pyramid a parallelepiped, and the planes by which each is cut be parallel to the base of the figure, the sections will not only be similar, but equal to each other: which forms a case of the Orthographic Projection.

PROP. III.

23. *Theorem.* The representation of any point in a given straight line not parallel to the Perspective Plane, is in the straight line joining the Vanishing Point of the given line, and that point in which the given line meets the Perspective Plane.

Let Q (fig. 3.) be any point in the given straight line QA, which meets the Perspective Plane PL in A; also, let O be the Point of View, and K the Vanishing Point of QA: Join K, A; the representation of Q is in KA.

For join OK and OQ; OK is parallel (art. 8.) to AQ, and therefore (E. Def. 35. 1.) OK is in the same plane as OA and Q, that is, (E. 2. 11.) in the plane of the triangle OAQ; therefore, both K and A are in the intersection of that plane, and PL; and the point q in which OQ meets the Perspective Plane, that is, the representation (art. 13.) of Q, is in the intersection of the same two planes PL and OAQ: But q cannot be further from A than K is; for then Oq could never meet AQ, however far it were produced toward Q; and it is manifest that q cannot be further from K than A is; therefore the representation of any point, Q, whatsoever, in QA, is in AK.

24. *Cor.* 1. The representation of any given portion of AQ is part of AK: Thus Aq is the representation of AQ, Ar that of AR, and rq that of RQ (art. 23. and 14.)

25. *Cor.* 2. The point q divides AK into two parts which have the ratio of OK to AQ: And if QA, OK, and KA be given, the distance of q, the representation of the given point Q, from A, may be found.

For the triangles OqK and AqQ (E. 15. and 29. 1. and 4. 6.) are similar.

$\therefore OK : Kq :: QA : Aq$

& $OK : QA :: Kq : Aq$ (E. 16. 5.)

$\therefore$ Ok, together with QA, *is to* QA, *as* KA *is to* Aq (E. 18. 5.); in which proportion the three first terms are given, and therefore (E. 12. 6.) the fourth term may be found.

26. *Cor.* 3. Hence, the points A and K being fixt, so that K continues to be the Vanishing Point of AQ, rq will continue to be the representation of an original of the same length, whilst QA and OK remain of the same length, whatever be the angle, at which they are both inclined to KA, and in whatever the same plane they are both situated: But if, the points r and q be fixt, rq remaining the same, whilst OK is varied in length, the original of rq, in AQ, will be varied as OK.

For draw rs parallel to OK or AQ; the two triangles OqK, and rqs will always be similar, on the supposition here made; and, therefore, rq and Kq remaining constant, rs will be varied as OK; but RQ has to rs the ratio of OR to Or, that is, the invariable ratio of KA to Kr; therefore, RQ is varied as OK.

27. *Cor.* 4. The representations of all straight lines, parallel to each other, but not parallel to the Perspective Plane, produced, if necessary, meet in the same point, which is the common Vanishing Point of all the original lines.

For the straight line drawn through the Point of View, parallel to any one of the given original lines, is parallel (E. 9. 11.) to all the rest; therefore, they have all the same Vanishing Point, and their representations, when produced, all pass (art. 24.) through that point.

28. *Cor.* 5. The Vanishing Point of all original straight lines which are perpendicular to the Perspective Plane, is the Center of the Picture.

29. *Cor.* 6. If the extremities of any number of

equal and parallel straight lines, lie in two parallel straight lines, which meet the Perspective Plane, their representations are bounded by two straight lines which meet in the Vanishing Point of the parallels containing their extremities.

For the representations of the two extremities of any of the lines, are in those of the lines which contain them; which latter representations meet (art. 27.) in the Vanishing Point of their originals.

PROP. IV.

30. *Problem.* The Point of View being given, to determine the representation of a given point, on a given Perspective Plane.

Let *C* (fig. 4.) be the Center of the Picture, and *A* the intersection of the Perspective Plane, and a straight line drawn perpendicular to it, from the given point, of which the representation is to be determined. Through *C* draw *CD* in any direction, and make *CD* (E. 3. 1.) equal to the Distance of the Picture; through *A* draw *AN* parallel (E. 31. 1.) to *CD*, but in a contrary direction, and make *AN* equal to the perpendicular distance of the given point from the Perspective Plane; join *DN* and *CA*; and let *DN* meet *CA* in *a*; *a* is the representation of the given point.

For, (art. 23. and 28.) the representation of the given point is in *CA*. But the two triangles *CaD*, *AaN* are equiangular (E. 15. and 29. 1.)

$\therefore DC : Ca :: NA : Aa$ (E. 4. 6.)

& $DC : NA :: Ca : Aa$ (E. 16. 5.)

$\therefore$ *DC*, together with *NA*, is to *NA*, as *CA* is to *Aa*

(E. 18. 5.); and, by the construction, *DC* is equal to the distance of the eye from the Vanishing Point (art. 28.) of a straight line in which the given point lies; and *NA* is equal to the distance of the given point, in that line, from the Perspective Plane: Therefore, (art. 25.) *a* is the representation of the given point.

31. *Cor.* The representation of a given straight line, which, being produced, does not pass through the Point of View, is the straight line joining the representations of its two extremities, (art. 14.) and may therefore be found by the preceding article.

32. *Remark.* A point in fixt space may be *given* according to several different specifications, from any one of which might be deduced the express data of the rest. Thus, the several perpendicular distances of a point from the floor, and two adjacent walls of a room, might be given; in which case the point is the common intersection of three given planes: The distances might be given of the point in fixt space from three given points, one in the floor, one in either of the two adjacent walls, and a third in the other wall; so that the point would be found to be the common intersection of three given spheres: Or, it might be designated by giving its distances from three given straight lines, when it would be one of the points of intersection of three given cylinders, and might be distinguished from the rest by certain conditions. From any one of these specifications might be deduced, though not always readily, the data of any other of them. In the common case of a vertical picture, the points to be drawn are usually described according to the first of the above methods; the three planes to which they

are referred, being the plane of the picture, another vertical plane perpendicular to it, and a third plane which is horizontal. They are, however, always immediately referred to the Perspective Plane, and are said to be given when their position with respect to that plane is determined. In art. 30, the point to be represented is *given* by its place, in a straight line, which meets the Perspective Plane at right angles in a given point, being supposed known. A point, however, of which the representation is sought, may not always be *expressly* so given; and some propositions are, therefore, added, which are applicable, when the given point is marked, by assigning its place, in a straight line not perpendicular to the Perspective Plane. In this case, something more must be specified than the inclination of the line, containing the given point, to the plane of the picture. For, although there can be but one straight line perpendicular to the same plane, at the same point, on the same side of it, (E. 13. 11.) yet all the straight lines in the surface of a right cone, of which that perpendicular is the axis, are inclined to the plane at the same angle. In the case abovementioned, therefore, the intersection of the Perspective Plane, and another given plane containing the given straight line, is further supposed to be given.

When the contrary is not specified, the Perspective Plane itself, and the Point of View, are always supposed to be given; and the Center, and the Distance, of the picture, are therefore understood to be known. By the Perspective Plane being given, is usually meant that its position is known relatively to two planes, one of which is parallel and the other perpendicular to the horizon; which two planes are in their nature fixt and determined by the plumb line.

PROP. V.

33. *Theorem.* The straight line which joins the Center of the Picture, and the Center of the Vanishing Line, of any Original Plane, is perpendicular to that Vanishing Line.

Let *PL* (fig. 1.) be the Perspective Plane, *O* the Point of View, *C* the Center, and *OC* the Distance of the picture, *xy* the Original Plane, *HL* its Base Line; *Dd* its Vanishing Line, *K* its Center, and *OK* its Distance. Join *K*, *C*; *KC* is perpendicular to *Dd*.

For *OC* (art. 4.) is perpendicular to the plane *PL*: Therefore, the plane *OKC* (E. 18. 11.) is perpendicular to *PL*, and any straight line in *PL*, drawn from *K*, which is perpendicular to *KC*, is also perpendicular (E. Def. 4. 11.) to the plane *OKC*, and therefore perpendicular (E. Def. 3. 11.) to *OK*: And there cannot be two different straight lines in *PL* meeting in *K*, and both perpendicular to *OK*; otherwise *OK* (E. 4. 11.) would be perpendicular to *PL*, which is contrary to the supposition: Therefore the straight line in *PL*, which is perpendicular to *OK*, at the point *K*, is also perpendicular to the plane *OKC*; but *Kd* is perpendicular to *OK*; (art. 11.) therefore *Kd*, or *Dd*, is perpendicular to the plane *OKC*, and consequently to the straight line *KC*.*

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* It is manifest, conversely, that if, from *C*, *CK* be drawn perpendicular to *Dd*, it shall meet *Dd* in a point *K*, which is the Center of

34. *Cor.* 1. The Principal Distance is a mean proportional between the distances of the Center of the Picture from the Center of the Vanishing Line of any plane, and from the Vanishing Point of all the straight lines which are perpendicular to that plane: And hence, when the first two distances are given, that Vanishing Point may be found. (E. 11. 6.)

In the plane OCK, draw Ok perpendicular to OK, and meeting KC produced in k. And because DK is perpendicular to OK and Kk, (art. 11. 33.) the plane Md is perpendicular (E. 4. and 18. 11.) to the plane OKk; Ok is therefore perpendicular to Md, and also (from the demonstration of E. 15. 11.) to XY; and therefore (E. 6. 11. and art. 8.) K is the Vanishing Point of the straight lines which are perpendicular to XY. But OC is a mean proportional between CK and Ck (E. cor. 8. 6).

35. *Cor.* 2. If either side of a right-angled triangle be equal to the Principal Distance; and the angle opposite to that side be equal to the inclination of a given plane to the plane of the picture, the subtending hypotenuse shall be equal to the Direct Distance of the Vanishing Line of that plane, and the remaining side shall be equal to the distance between the Center of the Vanishing Line and the Center of the Picture.

For such a triangle would be equal (E. 26. 1.) in every respect to the triangle OKC, it being manifest (E. 6. Def. 11.) that the angle OKC is the inclination of Md, or of its parallel (E. 15. 11.) XY, to PL.

the Vanishing Line Dd: Otherwise, two angles of a triangle are not less than two right angles, which is absurd (E. 17. 1.)

PROP. VI.

36. *Problem.* To determine the Vanishing Line of a given Original Plane; which meets the Perspective Plane, and also the Center, and the Direct Distance of that Vanishing Line.

Let *C* (fig. 5.) be the Principal Point, and *AB* the Base Line of the given plane. It is required to determine the Vanishing Line of that plane, its Center, and its Direct Distance.

Through *C* draw *KCE* perpendicular, (E. 12. 1.) and *Cd* parallel (E. 31. 1.) to *AB*. Make *Cd* equal (E. 3. 1.) to the Principal Distance, and draw *dm* parallel to *KCE;* at the point *d*, in *md*, make the angle *mdK*, equal (E. 23. 1.) to the inclination of the given Original Plane, to the plane of the picture; and let *dK* meet *KCE* in *K*. Through *K* draw *MKm* parallel to *AB*. *Mm* is the Vanishing Line of the given plane, *K* is its Center, and *Kd* is equal to its Direct Distance.

For the angle *dKC*, is equal to the alternate angle *Kdm*, (E. 29. 1.) and therefore equal to the given Original Plane's inclination; and *Cd* was made equal to the Principal Distance; and *KC* is perpendicular to *AB;* and therefore *K* is the Center, and *KD* the Direct Distance of the Vanishing Line: (art. 35.) Also, *MKm* was drawn parallel to *AB*, and is, therefore, (art. 10.) the Vanishing Line of the given plane.*

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* If it be required to determine the Vanishing Line, together with its Center and Direct Distance of another plane, the Base Line of which is given, and which is perpendicular to the given plane of the above Problem, let *AM* be the Base Line of the second plane; find

PROP. VII.

37. *Theorem.* The Vanishing Points of all straight lines situated in a given Original Plane, are in the Vanishing Line of that plane.

For the Vanishing Points (art. 8.) are determined by lines drawn through the Point of View, parallel to the original lines, and which are therefore in a plane passing through the Point of View parallel (E. 15. 11.) to the original plane; the intersections of these lines therefore with the Perspective Plane, are all of them in the Vanishing Line (art. 9.) of the Original Plane; and these intersections are the Vanishing Points.

38. *Cor.* 1. The Center of any Vanishing Line, is the Vanishing Point of all the Straight Lines in its Original Plane, which are perpendicular to the Base Line.

39. *Cor.* 2. If the representation of a parallelogram be given, the Vanishing Line of the Plane of its original may be found: For, let the opposite sides of the representation meet when produced;

(art. 34.) the Vanishing Point k of all straight lines which are perpendicular to the first plane, and through k draw $k\mu$ parallel to AM, and it shall be the Vanishing Line (art. 37 and 9.) required: And if from C, the straight line Cx be drawn perpendicular to $k\mu$, and meeting it in x, x is the Center of $k\mu$: Also, if $x\delta$ be made equal to Cd, and $C\delta$ be joined, $C\delta$ is equal to the Direct Distance of $k\mu$. If the intersection of the two planes be given, instead of the Base Line of the second plane, let $A\mu$ be its representation: Find k as before, and join $k\mu$; $k\mu$ is the Vanishing Line of the second plane, and its Center and Direct Distance may be found as in the former case. Also, if through A, AM be drawn parallel to $k\mu$, it will be the Base Line of that plane.

they will meet in the two Vanishing Points of each pair of parallels (art. 27.): The straight line, therefore, (art. 57.) joining these points, is the Vanishing Line of the Original Plane. But if two of the sides of the representation be parallel to each other, a straight line drawn parallel to either of them, through the point in which the other sides meet, is the Vanishing Line of the plane.

PROP. VIII.

40. *Theorem.* If either side of a right angled triangle, be equal to the Direct Distance of the Vanishing Line of a given Original Plane, and the opposite angle be equal to that made by any given original straight line, in the given plane, with its Base Line, the hypotenuse shall be equal to the distance of the Vanishing Point of the given line from the Point of View; and the remaining side shall be equal to the distance of that Vanishing Point from the Center of the Vanishing Line.

Let *AZ* (fig. 1.) be any straight line in the original plane *XY*, and let *V* be its Vanishing Point, the figure being constructed as in art. 33. Join *OV*. Then since *OV* (art. 8.) is parallel to *AZ*, and *VK* to *HL*, (art. 10.) the angle *OVK* is equal (E. 10. 11.) to the angle *HAZ*: And *OKV* is a right angle, (art. 11.) and *OK* is the Direct Distance of the Vanishing Line, and *VK* the distance of the Center of that line from the Vanishing Point of *AZ*: Therefore, the triangle described in the Theorem, is equal, in every respect, to the triangle *OKV*.

41. *Cor.* The other acute angle of the triangle is

equal (E. 26. 8.) to the angle in the given Plane, made by the original line, with a straight line perpendicular to the Base Line. Also, if the two sides containing the right angle of a right-angled triangle, be equal respectively to the Direct Distance of the Vanishing Line of any given plane, and the distance between the Center of that line and the Vanishing Point of any original line in the given plane, the angle opposite to the side which is equal to the Direct Distance, is equal to that which the original line makes with the Base Line of the given plane.

PROP. IX.

42. *Problem.* To determine the Vanishing Point of a given straight line, situated in a given plane which meets the Perspective Plane.

Let *AB* (fig. 6.) be the Base Line of the given plane, and *B* the point in which the given line meets it: Find (art. 36.) the Vanishing Line *KM* of the given plane, *K* its Center, and *Kd* its Direct Distance: From the point *K*, draw *KF* perpendicular (E. 11. 1.) to *KM*, and make *KF* equal (E. 3. 1.) to *Kd*. From the point *F*, draw the straight line *Fb*, (Dem. of art. 36.) making with *KM*, the angle *FbM* equal to the angle which the given straight line makes with the Base of the Original Plane, and on the same side of *FK*, that the given line is, with respect to a straight line in the original plane, perpendicular to *AB* at *B*: Then is *b* the Vanishing Point of the given straight line.

For the Vanishing Point is in *bM*, (art. 37.) and it is at the distance *Kb* from *K* (art. 40.) and it is on that side of *FK* toward which *Fb* is drawn. (Art. 8.)

PROP. X.

43. *Problem.* From a given point in the representation of a given straight line, to draw the representation of another given straight line, which makes, with the former, a given angle, the two lines lying in a given plane, which meets the Perspective Plane.

Let *AB* (fig. 6. constructed as in prop. 9.) be the Base Line of the plane in which the two original lines lie, and *gb* the representation given, and *g* the given point in it: It is required to draw from *g*, *gM*, such that it shall be the representation of a straight line, which makes, with the original of *gb*, a given angle.

At the point *b*, make the angle *FbZ*, equal (E. 23. 1.) to the given angle, and through *F* draw *FM* parallel (E. 31. 1.) to *Zb*, and join *gM*: *gM* is the representation sought.

For since *FbM* is equal (art. 41.) to the angle made by the original of *gb* with the Base Line, and *FbZ* is equal to the angle contained by the two original lines, *Zbm* is equal to the angle made by the other line with the Base Line. But the angle *FMb*, is equal (E. 29. 1.) to the angle *Zbm*, and, therefore, (art. 42.) *M* is the Vanishing Point of the line, of which the representation is sought: Therefore *GM* is that representation.*

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* If it be required to draw from any given point *i*, the representation of a straight line, parallel to the original of *gh*, and lying in

44. *Cor.* The angle *bFM*, which is equal (art. 41.) to the sum, or the difference of the angles made by the two given original lines, with a straight line drawn from the angular point perpendicular to the base line, is also equal to the given angle, and regard must be had to this, when the representation given of one of the lines is parallel to the Base Line.

PROP. XI.

45. *Problem.* To cut off from the representation of a given indefinite straight line, situated in a given plane, a part which shall be the representation of a given part of that line.

Let *Bb* (fig. 6.) be the representation of the given indefinite line, *B* being the point in which it meets the Base Line, *AB*, of the given plane containing it. It is required to cut off from *Bb*, a part which shall be the representation of a given part of the original of *Bb*.

Find (art. 36.) the Vanishing Line *Mm* of the given plane, its Center *K*, and its Direct Distance *Kd*; draw (E. 11. 1.) *KF* perpendicular to *mM*, and make it equal (E. 3. 1.) to *Kd*; let *Bb* meet *mM* in *b*; which is therefore (art. 37.) the Vanishing Point of the Original of *Bb*; join *bF*: At the point *B* in *AB*, make the angle *ABH* equal (E. 23. 1.) to the angle *FbM*, *BG* equal (E. 3. 1.) to the distance of

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the same given plane, produce *gk* to meet *mM* in *b*, and join *i*, *b*; *ib* is the representation sought (art. 27 and 37.). It is manifest, that, if by the measurement of a base line, and a series of angles, as is taught in the Elements of Trigonometry, a survey had been taken of a plain, shewing the places of any remarkable objects standing upon it, the same data would enable us, by the help of this problem, very readily to draw the perspective delineation of it.

the nearer extremity of the given part from B, and GH equal to the given part: Join F,G, and F,H; and let FG meet Bb in g, and FH meet it in h; gh is the representation sought.

For the two triangles Fgb, and BgG, are equiangular, (E. 15 and 29. 1.) and their homologous sides are proportionals (E. 4. 6.): Also, Fb is equal to the distance between the Point of View and the Vanishing Point of the given line, (art. 40. and 41.) and BG is equal to the distance of the nearer extremity of the given part from its intersection with the plane of the picture: Therefore, (art. 25.) g is the representation of that extremity. In the same manner h may be shewn to be the representation of its other extremity: Therefore, (art. 14.) gh is the representation sought.*

46. *Cor.* Thus the representation may be determined of a given point in a given straight line, which is inclined at any given angle to the Perspective Plane.

PROP. XII.

47. *Problem.* To determine the representation of a point which divides, in any given ratio, a given finite straight line, of which the representation and the Vanishing Point are also given.

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* It is manifest, from the demonstration, as also from art. 26, that that the point g might also be determined, by making $B\gamma$ on the Base Line of the given Plane equal to BG, and $b\phi$ on the Vanishing Line equal to bF, and then joining $\phi\gamma$. If the original line were perpendicular to AB, hg produced, would pass through K; and this construction would agree with that in art. 30; which proposition might have been deduced from this 11th proposition: But it is given separately on account of its ready application in practical Perspective.

Let ab (fig. 7.) be the representation of the given finite straight line, and K its Vanishing Point: It is required to find a point in ab, the original of which divides the original of ab in a given ratio.

If ab be produced, (art. 24.) it shall pass through K; produce it to K, and through K draw any straight line KP; on the contrary side of aK, through any point 2, in Ka, produced, below a, draw $2LM$ parallel (E. 31. 1.) to KP; join P,a and P,b, and let Pa and Pb produced, meet $2M$ in L and M; divide (E. 10. 6.) LM in N, so that LN, may have to NM, the given ratio; join P, N, and let PN cut ab in c. Then is c the point required.

For let E be the point in which the original line, produced, if necessary, meets the Perspective Plane; E either coincides with a, or is in ba produced (art. 23. and 24.): Make KO equal (E. 3. 1.) to the distance between the Point of View, and the given point K, and through E draw EBm parallel to KO, or $2M$; join O,a, O,c, and O,b, and let Oa, Oc, Ob, produced, if necessary, meet Em in A, C, and B respectively; and let PL, PN, PM, produced if necessary, meet Em, in l, n and m respectively. Then (art. 25 and 28.) AB is equal to the original of ab, AC to that of ac, and CB to that of cb: Also, lm is equal to the original of ab, seen from a point in the same line with K and the first Point View, and at the distance PK from K; and ln is equal to the original of ac, and nm to that of cb.

But (art. 26.) $AC : ln :: OK : PK$
& $CB : nm :: OK : PK$
$\therefore AC : ln :: CB : nm$ (E. 11. 5.)
$\therefore AC : CB :: ln : nm$ (E. 16. 5.)

Again, the triangle *PNL*, is equiangular to the triangle *Pnl*, and *PNM* to *Pnm*.

$\therefore PN : NL :: Pn : nl$ (E. 4. 6.)
& $PN : Pn :: NL : nl$
In the same manner $PN : Pn :: NM : nm$
$\therefore NL : nl :: NM : nm$
$\therefore NL : NM :: nl : nm$
But $Ac : cB :: nl : nm$
$\therefore AC : CB :: NL : NM$

And therefore *c* is the representation of a point *C*, which divides the given straight line in the given ratio.

PROP. XIII.

48. *Problem.* To determine the representation of a given finite straight line, parallel to the Perspective Plane.

Find (art. 30.) the representation *a*, (fig. 4. the construction remaining as in art. 30.) of one extremity of the given line, and let *AB* be the Base Line of a plane containing the given line, and passing through *A*; i. e. meeting the Perspective Plane at right angles; make *AB* equal to the given original, (E. 3. 1.) and join *C*, *B*; through *a* draw *ab* parallel (E. 31. 1.) to *AB*, and meeting *CB* in *b*. Then is *ab* the representation of the given line.

For the given straight line, by the hypothesis, can never meet the Perspective Plane, and therefore can never meet *AB*; *AB*, being in the same plane with it, is, therefore, parallel to it, and its representation (art. 17.) is parallel to *AB*.

Again, the two triangles *CAB*, *Cab* are equiangular.

$\therefore BA : AC :: ba : aC$ (E. 4. 6.)
$\therefore BA : ba :: AC : aC$ (E. 16. 5.)

And it may be shewn, as in art. 30, that *AC* is to *aC*, as *DC*, together with *AN*, is to *DC*: But *DC* is the Principal Distance, and *DC* together with *AN*, is equal to the perpendicular distance of the eye from the plane containing the given line: Therefore, (art. 16. and 19.) *ab* is the representation of that given line.

49. *Cor.* The length of the representation may be determined by taking any point *H* in *CD*, and any point *G* in *AN*, drawing from *a*, *ag* parallel to *DC*, making *GF*, in any straight line drawn from *G*, equal to the given original line, joining *H*,*G*, *H*,*F*, and drawing *gf* (E. 31. 1.) parallel to *GF*, *g* and *f* being the points in which *HG* and *HF* meet *ag* and *gf*; *gf* is equal in length to the representation of the given straight line.

For from the points *C* and *H*, draw (E. 12. 1.) the straight lines *CLM* and *HIK* each perpendicular to *AM*, meeting *Lx* in *L* and *I* respectively: The angles at *I* and *L* (E. 29. 1.) are right angles, and *CM* is parallel (E. 28. 1.) to *HK*: Therefore, *CK* and *CI* are parallelograms, and (E. 34. 1.) *CM* is equal to *HK*, and *CL* to *HI*. Again, (E. 29. 1. and 4. 6.) the two triangles *GHK* and *gHI*, as also the two *ACM* and *aCL*, and the two *HGF* and *Hgf*, are equiangular respectively;

$\therefore HC : Hg :: HK \text{ or } CM : HI \text{ or } CL$
$\& \ CM : CL :: CA : Ca$
$\therefore HG : Hg :: CA : Ca$ (E. 11. 5.)
But $HG : Hg :: GF : gf$
$\& \ CA : Ca :: AB : ab$
$\therefore GF : gf :: AB : ab$ (E. 11. 5.)
$\& \ GF : AB :: gf : ab$ (E. 16. 5.)

But *GF* is equal to *AB* by the construction; therefore *gf* is equal to *ab*, which (art. 48.) was shewn to be the representation sought.

50. *Cor.* 2. Hence, also, if the inclination be known of the given original parallel straight line to any plane perpendicular to the Perspective Plane, the Vanishing Line of which is *DC*, its representation may be drawn, after finding the point *a*; by making the angle *bad* equal to that inclination, and cutting off *ba* equal to *gf*, determined as before (E. 5. Def. 11. art. 21.). If* the inclination of several straight lines, parallel to each other, and to the Perspective Plane, and all making the same angle with the plane of which *CD* is the Vanishing Line, be given, and it be required to draw a straight line which shall be parallel (art. 18.) to their representations; take any point *d*, draw *dx* parallel to *CD*, and *dy*, making the angle *ydx* equal to the given inclination, and *yd* is the line required.

Scholium.

The preceding articles contain the Elements of Linear Perspective. To the representation of objects, terminated by straight lines, and plane surfaces, they are readily applicable; and this is, perhaps, the only case in which Perspective is actually employed, with strictness, by artists. When the object is a plane figure, but not contained by straight lines, it is sufficient to determine several points in its representation, and through these, to draw by the hand, a curve line. But if the object be a solid, the surface of which is not plane, its visible outline must be previously determined, which is manifestly the locus of the points in which straight lines drawn from the Point of View touch the body.

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* If several straight lines be parallel to each other, they shall have the same inclination to any the same plane which they all meet. For let two points be taken, one in each of any two of the parallels equidistant from the plane: Join the points in which two perpendiculars drawn (E. 11. 11.) from them meet the plane, and the points in which the two parallels meet the same plane: Then the angles, contained by the two parallel lines first taken, and the two perpendiculars respectively, are equal (E. 6. and 10. 11.). Therefore, the remaining two acute angles in each of the two right-angled triangles, which (E. 5. Def. 11.) measure the inclination of the two parallels, are also equal (E. 32. 1.). The 8th proposition of the 11th Book of Euclid's Elements, is a particular case of this general theorem.

It does not enter the plan of an elementary work to give instances belonging to this last case: otherwise the example of the cone might have been taken, as affording a striking specimen of elegant geometrical construction. It may be seen in Hamilton's Perspective.

In all cases, previously to any delineation, the relative positions of the Point of View, the Perspective Plane, and of the Original Object, must be known.

In drawing from nature, the Perspective Plane is usually supposed to be perpendicular to the horizon; because then the representations of vertical objects, are, themselves, vertical; and such a Point of View ought to be chosen, that the objects to be drawn may be seen from it as completely as is possible; i. e. so that the most interesting parts may not cover each other, nor be too much diminished by distance. In general, the height of the eye above the horizon should not be less than one third, nor greater than one half, of the depth of the picture; the Principal Distance should be about one third part greater than the height of the eye: If the Point of View be fixt, the utmost limits of the prospect, which can be delineated from it, both in breadth and height, are determined by two straight lines subtending right angles at that Point of View; and conversely if the Original Object be given, the Principal Distance is, in some measure, thereby limited; because the Point of View must be found in the arch of a semicircle, described upon the breadth of the object as a diameter; or if it be further removed from the object, it must be at such a distance from it, that all its parts may be seen distinctly. Again, as we usually look straight forwards in the direction of a line parallel to the horizon, it is manifest, in the case of the vertical Perspective Plane, that the lowest point of the object ought not to have an angle of depression greater than 45° below this line; nor ought the highest point to have an angle of elevation greater than 45° above it. Hence, the Original Object being given, the proper height of the eye becomes limited. The Center of the Picture is, commonly, equally remote from the two extremities of the drawing, unless when it is intended that particular sides of a view should strike the eye more forcibly than the rest.

It is further to be remarked, that the lines immediately operated upon are usually less than the lines given, but each containing the same *number* of equal parts of a certain magnitude, as the line for which it is taken contains of another certain magnitude. This is effected by employing the same scale of equal parts throughout the same delineation; and the size of the scale is regulated by the dimensions which it is desired the picture should have.—The Base Line of any plane, is at once the original and the picture of that line, and the parts of the one are equal to the parts of the other: They have, therefore, a common scale. Agreeably to these observations, it is convenient, before any delineation is begun, to prepare the Perspective Plane in the following manner:

Let *AB* (fig. 8.) be the Base Line, and *Dd* the Vanishing Line of the plane upon which the objects, to be delineated, stand, and which is here supposed to be the Ground Plane; let *C* be the Center of the Picture; *CD*, *Cd*, and *CF*, which is drawn perpendicular to *Dd*, each equal to the Principal Distance. Divide *AB* in the points 1, 2, 3, &c. into equal parts, according to the scale upon which the drawing is to be made; join *CA*; also, join *D*,1, *D*,2, *D*,3, &c. and let *D*1, *D*2, *D*3, &c. meet* *CA* in the points 1, 2, 3, &c. The position of any point on the ground plane is known, by its perpendicular distances being given from *AB*, and from another given straight line at right angles to *AB*, which it is best to take as drawn from the point *L*, where *FC* produced meets *AB*, and which may be called the Station Line. If then the distance of the given point from the Base Line be set off from *A*, on *AC*, its representation (art. 30) will be in a straight line, drawn from that division, parallel to the Base Line; and if its distance from the Station Line be set off from *L*, on *LA*, or *LB*, its representation will also lie in a straight line drawn from that division to *C*; it will therefore be in the intersection of those two lines, and is determined.

Also, if there may be many angles on the ground plane, which are to be represented, straight lines must be drawn from *F*, meeting *Dd* in 10, 20, 30, &c. and making with *FC*, (art. 43. and 44.) angles equal to given rectilineal angles, the second double of the first, the third triple of the first, and so on. These divisions will greatly facilitate the subsequent delineation: And in cases where points are to be found, by drawing straight lines which are not afterwards wanted, it is better to determine such points by stretching a thread or a hair, than to load the picture unnecessarily with marks. The scale marked on *Dd*, will serve for the delineation of all *angles*, situated in any planes which are parallel to the ground plane. But if there be many *lines* to be represented which lie in any horizontal surface, the scale *AB*, must be marked on the Base Line of that surface. The work will then proceed as before. The same method of delineation may be applied to any given plane, which is perpendicular to the Perspective Plane, although not parallel to the horizon, if the two scales be marked, in the same manner, upon its Base and Vanishing Lines: The Vanishing Line, and its Center, of the given plane, being previously determined, by art. 36.

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* When, from the dimensions of the picture, these lines cannot be conveniently drawn, the divisions of *AC*, may be calculated from art. 25.

When the picture of a given straight line perpendicular to the Ground Plane is sought, the representation of the point in which it meets that plane must first be determined: If then *BZ* be drawn at right angles to *AB*, and the length of the original be set off upon it from *B*, the length of the representation required is found by art. 50, and its direction is perpendicular to *AB*.

Examples.

1. *E* (fig. 8.) is the representation of a cube, contained by squares of which the sides are each

two equal parts of the assumed scale in length, and standing upon an oblong portion of tesselated pavement, composed of squares, of which the sides are each one of those parts. The breadth of the pavement is four parts, and its length six. The two shorter sides of the oblong are parallel to *AB*, and the nearest is one part from *AB*; the two longer sides are perpendicular to *AB*, and the nearest of them is six parts from the Station Line: The nearest side of the base of the cube is five parts from *AB*, and the side nearest to the Station Line is distant from it seven parts. The pavement is drawn by means of art. 30, and the height of the representation of the cube, is most easily determined from art. 20.

2. Let *fg*, (fig. 8.) which is bisected by *CL*, and parallel to *AB*, be the representation of the side of a regular hexagon, lying on the ground plane, parallel to the Base Line, and let it be required to draw the representations of the remaining sides. Join *fC*, *gC*, and let the straight lines *F* 30, and *F* 30 on each side of *FC*, make, with *FC*, an angle equal to that which the side adjacent to the given side makes with a perpendicular to the Base Line: Join *f*, 30, and *g*, 30, *f* 30' and *g* 30'; let *f* 30 and *g* 30', meet in *x*; through *x* draw *lxh* parallel to *fg*, and meeting *g* 30' and *f* 30', in *h* and *l*; also, let *f* 30, meet *Cg* in *i*, and *g* 30' meet *Cf* in *k*; join *lk*, *k*,*i* and *i*,*h* : *fghikl* is the representation required. See art. 43 and 44.

3. Let *pq* (fig. 8.) be the representation of the chord of a given arch of a circle on the ground plane, and let it be required to find the representation of the whole circumference. Draw *qr*, meeting *Dd* in any point *r*; join *Fr*, and make the angle *rFs* equal to that contained in the segment cut off by the ori-

ginal of *pq*, drawing *Fs* so that *s* and *p* may be on different sides of *qr*; join *s*,*p*, and let *sp* meet *qr* in *t*; *t* is a point in the representation of the circumference of the circle (E. 21. 1. and art. 44.); and thus may any number of points of the representation required be found. If *Dd* should be found too short for the operations required in this case, after any other point, *u*, of the representation has been determined, *tu* may be joined, and the process continued from *tu*, as it was carried on from *pq*.

4. Let it be required to represent a given circle lying on the ground plane.

Take the point *O* (fig. 8.) at the same perpendicular distances from the Base and Station Lines as the center of the given circle; and from *O* as a Center, at a distance equal to the radius of the given circle describe the circle *MN*2. From any point *M*, in the circumference, draw the straight line *MN*, which produce until it meet *AB* in *P*: Join *FM*, and draw *FV* parallel to *PNM*, meeting *Dd* in *V*; join *P*, *V*, and let *PV* meet *FM* in *m*; then is *m* the representation (art. 46.) of *M*, and thus may any number of points be determined of the representation required.

The representation of a circle on the Ground Plane, (art. 22.) will, according to the Principles of Conic Sections, be in all cases an ellipse, unless the Perspective Plane cut the cone of rays in what has been called subcontrary position: In this case the representation of the circle will be itself a circle. If a given circle, situate on the Ground Plane, touch the Perspective Plane in that point in which the Base of the Ground Plane is met by the Station Line, and the height of the eye be also given, from these data, the Distance of the Picture may be determined, by a geometrical construction, such that the representation shall be a circle.

5. Let it be required to delineate a parallelepiped, standing upon a square base, of which *GH* (fig. 8. No. 2.) represents one of the sides; the height of the solid being seven feet, and one of the diameters of its base being parallel to the Ground Line.

Draw *DA* perpendicular to *AB:* Join *Hd,* and through *G* draw *GI* parallel to *AB,* meeting *Hd* in *I.* Also, join *ID* and *Gd,* and let them cut each other in *K.* Then does *GHIK* represent the square base of the original figure. Take any point *R* in *Dd;* join *RA* and *R*7, *A*7 containing seven equal parts of the scale upon which the drawing is made. Draw *Ha* parallel to *AB,* meeting *RA* in *a,* and *ab* parallel to *AD,* meeting *R*7 in *b.* From the points *G,H* and *I,* draw *Gg, Hh, Ii,* all parallel to *AD,* and *bh* parallel to *AB,* meeting *Hh* in *h.* Join *hD* and *hd,* and let *hD* meet *Gg* in *g,* and *hd* meet *Ii* in *i.* Lastly, join *gd* in *iD,* and let them cut each other in *k.* Then is *GghHIikg* the representation required.

6. Let it be required to delineate a pyramid standing upon a square base, of which *NO* (fig. 8. No. 2.) represents one of the sides; *NO* being parallel to *AB,* and the height of the pyramid ten feet.

Join *NC, Nd, OC,* and let *Nd* meet *OC* in *P;* Draw *PM* parallel to *AB,* meeting *NC* in *M.* *NOPM* represents the base of the pyramid. Join *NP, OM,* and let them meet in *Q.* Draw *Qq* perpendicular to *AB,* and determine its length, in the same manner as the length of *Hh* was determined, in the preceding example. Join *qM, qN, qO.* Then is *qMNOq* the representation sought.

7. In fig. 9, is represented the interior of a room the breadth of which is 16 feet, the length 24 feet, and the height 12 feet; the height of its door is 8 feet, and the width 3 feet, the thickness of its wall is one foot: The height of the eye is 5 feet, and the Principal Distance is here taken 10 feet 6 inches.

Remark. The last example is intended to shew how very insufficient a *mere delineation,* however accurate, is to excite the idea of the original object, the outlines of which it represents. We judge of visible objects, not only by the *direction* of the rays proceeding from their several parts, which the preceding propositions serve to determine, but also from the masses of light and shade, the contrasts of colour, and the gradations of tint, which we observe them to exhibit. Every part of the representation of the chamber which does not receive the sun's rays directly, and is only visible by means of reflected light, ought to be shaded in order to give the picture resemblance. The doctrine of light and shade, which belongs to Aereal Perspective, or the Chiaroscuro, as it has been termed by the Italians, is investigated upon mathematical principles, by Lambert in his Photometria; and this is almost the only attempt which has been made to treat that subject scientifically. Dr. Brook Taylor, to the second edition of his Linear Perspective, added an appendix, containing—A New Theory for Mixing Colours, taken from Sir Isaac Newton's Opticks: And Mr. Sowerby has lately published a work on the Colouring of subjects belonging to Natural History.

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SECTION II.

On the Examination of Pictures, supposed to be drawn according to the Principles of Linear Perspective.

PROP. I.

51. *Problem.* The representation being given of a straight line, situated in a given plane, together with the Distance of the Picture, to determine the position and the length of its original.

Let *hg* (fig. 6.) represent a straight line situated in a given plane, of which *AB* is the Base Line: It is required to determine the position and the length of the original of *hg*.

Find (art. 36.) *mM* the Vanishing Line of the given plane, together with its Center *K*, and its Direct Distance: Draw (E. 11. 1.) *KF* perpendicular to *mM*, and make it equal (E. 3. 1.) to the Direct Distance of *mM*; produce *hg*, both ways, to meet *mM* in *b* and *AB* in *B*; join *bF*: Then is the angle *FbM* equal (art. 41.) to that which the original line makes with *AB*, and *B* is manifestly the point in which the Original meets *AB*: Its position is therefore determined. Again, from the point *B* draw (E. 31. 1.) *BH* parallel to *bF*; join *Fg* and

Fh; and let *Fg* and *Fh* when produced, meet *BH* in *G* and *H*: *GH* is equal in length to the original of *gh*. This is evident from art. 25 and 26.

52. *Cor.* Hence, from the representation of a point situated in a given plane, may be determined the place of that point.

For let *a* (fig. 4.) be the representation, and *GA* the Base Line given, *CD* the Vanishing Line, and *C* its Center found as before: Make *CD* equal to its Direct Distance: Join *Ca* and *Da*, and let *Ca*, *Da*, when produced, meet *GA* in *A* and *N*: Then is *AN* (note to art. 45. and art. 51.) equal to the original of *Aa*, i. e. to the perpendicular distance of the original of *a* from the point A, on the given plane. And thus, from the representation of a curve, known to be situated on a given plane, might be determined the curve itself, by finding a sufficient number of its points.

PROP. II.

53. *Problem.* The representation and the Vanishing Point of a divided straight line being given, to determine the ratio of the parts of the original line.

Let *ab* (fig. 7.) divided in *c*, be the representation, and *K* the Vanishing Point of the divided original straight line: It is required to determine the ratio of the original of *ac*, to the original of *cb*.

Through *K*, in any direction draw *KP*, and also from any point 2, in *ba* produced, draw 2*LM* parallel (E. 31. 1.) to *KP*; from any point *P* in *KP*, draw through *a*, *c*, *b*, *PaL*, *PcN*, *PbM*, meeting 2*LM*, in *L*, *N*, and *M*: The original of *ac* is to the original of *cb*, as *LN* is to *MM*.

For let KO be taken, in KP, equal to the distance of the Point of View from K; from E, the point in which the original line meets the Perspective Plane, let EB be drawn parallel to KP; join Oa, Oc, Ob, and let Oa, Oc, Ob meet EB in A, C, B respectively, then (art. 26.) is AB equal to the original line, and similarly divided in C; and it may be shewn, as in art. 47, that $LN : NM :: AC : CB$.

54. *Cor.* If the representation acb, and the ratio of the parts of the original line had been given, its Vanishing Point might have been found, by drawing through c any straight line dcf, making dc to cf, as the original of ac to the original of cb, and joining a,d and f,b; ad and fb produced, would meet in a point P, from which, if a straight line PK were drawn parallel to dcf, it would meet ab produced in K the Vanishing Point of the original of ab: Further, if GH represent any straight line perpendicular to a plane containing the original of ab, and KI be drawn at right angles to GH, KI is the Vanishing Line of that plane.

55. *Remark.* The Vanishing Point of a divided straight line may often readily be found by art. 27. If, for example, the side of a house not parallel to the Perspective Plane, be represented, it is only necessary to produce two of the lines which bound the representation until they meet. The relative breadths and situations of the doors and windows, may then be found by art. 53: But if a divided straight line in the picture have no Vanishing Point, the parts of its original have (art. 19.) to each other the same ratio as their representations.

PROP. III.

56. *Problem.* The representation being given, of an angle situated in a given plane, together with the Distance of the Picture, to find an angle equal to the original of that representation.

Let *hgl* (fig. 6.) be the representation of an angle situated in a given plane, of which *EB* is the Base Line: It is required to find an angle equal to the original of *hgl*.

Find (art. 36.) *mM* the Vanishing Line of the given plane, together with its Center *K*, and its Direct Distance. Produce *gh* and *gl* to meet *mM* in *b* and *M*; draw (E. 11. 1.) *KF* perpendicular to *mM*, and make it equal (E. 3. 1.) to the Direct Distance of *Mm*; join *bF*, *MF*; the angle *bFM* is equal to the original of *hgl*.

For (art. 37.) *b* and *M* are the Vanishing Points of the originals of *gL* and *gl*; and therefore (art. 8. and *E.* 10. 11.) the angle subtended by *bM* at the Point of View, is equal to the original of *hgl*: But since, by the construction, *Kb* and *KM* are the distances of *b* and *M* from the Center of *Mm*, and *KF* is at right angles to *Mm*, and equal to its Direct Distance, *Fb* and *FM* are equal to the distances of *b* and *M*, respectively, from the Point of View; and therefore the triangle *bFM* is equal (E. 8. 1.) to that contained by *bM*, and the straight lines drawn from *b* and *M* to the Point of View, and the angle *bFM* is equal to the original of *hgl*.

57. *Cor.* 1. Hence the data being the same as in the problem, if two straight lines meet in the picture, a straight line may be drawn through a given point

of the one, so as to cut off from them parts, the originals of which are equal.

For let it be required to draw through *h*, a straight line to cut off from *gM*, a part *gl*, the original of which is equal to the original of *gh*. Find the angle *bFM* as before, and produce *bF* to *X*: Bisect (E. 9. 1.) the angle *MFX* by the straight line *FY*. From the point *h* draw (art. 43.) *hl*, such that the original of *ghl* may be equal to the angle *MFY* or *YFX*, and let it meet *gM* in *l*: The original of *gl* is equal to that of *gh*.

For *hgl* represents a triangle, of which the exterior angle is, by the construction, double one of the interior and opposite angles; but (E. 32. 1.) it is also equal to the sum of the two interior and opposite angles: These angles are, therefore, equal to each other, and (E. 6. 1.) the original of *gh*, is, therefore, equal to the original of *gl*.

Or, join *h*, and the point in which *FY* meets, when produced, the Vanishing Line, *Mm*, by a straight line, cutting *gM* in *l*; the original of *gl* is equal to the original of *gh*.

For the point in which *FY* produced, meets *Mm*, is (art. 44. and 37.) the Vanishing Point of the straight line, which bisects the original of the angle *lgB*, and therefore, also, of the parallel (art. 27.) to that straight line, drawn from the original of *h*, which parallel makes the original of *gh* equal to the original of *gl*. (E. 29. and 6. 1.)

58. *Cor.* 2. If from any other point *e* in *gh* a straight line be drawn to the point in which *FY* produced meets *Mm*, it shall cut *gl* in *f*, so that the original of *gf* is equal to the original of *ge*.

59. *Remark.* The preceding article may often serve to facilitate the practice of Perspective. For

if *gh* (fig. 6. No. 2.) be the Base Line containing the graduated scale, it is obvious that the representation *gf*, of any required part of the original of *gl*, may thus be determined; or *gl* may be divided so that the originals of its parts shall be all equal. If, for example, the side of a building is to be delineated, the places and the breadths of the doors and windows may, by this method, be readily found. The point *Y* in which *FY* meets *mM* may, in this case, be determined, by making *MY* equal to *MF*: For if *FY* be joined, the straight line *FY* bisects the angle *MFX* (E. 5. and 29. 1.) *FX* being parallel to *Mm*. *FM* is manifestly equal to the distance between the Vanishing Point of the original of *gl* and the Point of View. When the side of a building to be delineated, is perpendicular to the Perspective Plane, its representation tends to the Center *K*, and the point in *mM*, from which the straight lines are to be drawn to the scale on *AB*, is at the same distance from *K*, as the Point of View, agreeably to what was shewn in art. 30.

In (fig. 14.) an example is given of this mode of delineating. The height of the eye *LK*, is here supposed to be 12 feet, and the Principal Distance *KO*, 42 feet, to which *KD* and *kd* are, each of them, made equal. The house *ABc* has one of its sides *Ab*, of which only half is seen, in the plane of the picture, and the other *Bc* perpendicular to it. *Bb* represents an height of 22 feet, and *BC* a length of 38 feet. The breadth of each window and also of the door is four feet, the distance of the first window in *Bc* from *Bb* is 3 feet, which also measures the interval between any two windows, except those two which are separated by the door: The height of the lower part of the lower windows from the ground is

3 feet, and that of the upper part 11 feet. *BP* being, therefore, taken equal to three equal parts of the scale marked on the Base Line, and *B*2 to eleven of those parts, the lower windows lie between the two straight lines joining *PK* and *2K:* And their breadths and places are marked on *BC* by straight lines drawn from *D*. Both sides of the house *gEf* are inclined to the Perspective Plane, at an angle equal to half a right angle, and therefore *EG* and *EF* produced, meet *Dd* in the points *Dd*. Join *dO*, and from *d* and *D*, set off *dM* and *DN*, each equal to *dO*. *EF* may then be divided by straight lines drawn from *M*, and *Eg* by straight lines drawn from *N* to the base line. The length of each side is there supposed to be 20 feet, the height represented by *Ee* 16 feet, and that of a straight line drawn from the summit of the roof, perpendicular to the ground, which is represented by *hH*, is 21 feet. The dotted lines are left to shew the method of determining the several points, by the joining of which the representation is completed.

PROP. IV.

60. *Problem.* The representation of a triangle, of which the angles are equal to three given rectilineal angles, and the Vanishing Line of its plane being given, to find the Center of the Vanishing Line and its Direct Distance.

Let *acb* (fig. 10.) be the given representation of the triangle, and *LN* the Vanishing Line of its plane: It is required to find the Center of *LN*, and its Direct Distance.

First, let one of the sides of the representation, as *bc*, be parallel to *LN*; produce *ac* and *ab* to meet *LN* in *M* and *N*, and upon *MN* describe (E. 33. 3.) a segment of a circle *MON*, capable of containing an angle equal to that which is included by the originals of *ac* and *ab*; at the point *M* in *NM*, make the angle *NMO* equal (E. 23. 1.) to that made by the original of *ac* with the original of *cb*, and let *MO* meet the circumference *MON* in *O*; draw *OK* perpendicular (E. 12. 1.) to *LN*; then is *K* the Center of *LN*, and *OK* is equal to its Direct Distance.

For, join *ON*; the angle *MON* is equal (E. 21. 3.) to the original of *bac*, and *ONM* (E. 32. 1.) to that of *abc*; but the original of *bc* is parallel (art. 17. and 10. and E. 9. 11.) to the Base Line, and therefore the angles, which the other sides of the triangle make with it, are equal (E. 29. 1.) to the angles which they make with the Base Line; Therefore the angles *OMN* and *MNO*, are equal to those which the sides containing the original of the angle *bac* make with the Base Line: If therefore *K* were the Center of *LN*, and *KO* its Direct Distance, *acb* would be the representation (art. 43. and 44.) of the original triangle; and it cannot be the representation of that triangle upon any other supposition; for *MN* must subtend an angle equal to the original of *bac*, and the angle *NMO* must be equal to the original of *acb*: Therefore, *K* is the Center, and *KO* the distance required.

But, if *bc* be not parallel to *LN*, let it, when produced, meet *LN* in *L*; upon *LM* describe a segment of a circle *LOM*, capable of containing an angle equal to the original of *acb*, and let its circumference meet *MON*, described as before, in *O*. From *O* draw *OK* perpendicular to *LN*; and if *OL*, *OM*, and *ON* be joined, it may be proved, as in the first case, that *K* is the Center of the Vanishing Line, and *KO* the distance of *K* from the Point of View.

PROP. V.

61. *Problem.* The representation being given of a parallelogram, which is similar to a given parallelogram, to determine the Vanishing Line of its plane, the Center of that Vanishing Line, and its Direct Distance.

Let *dfgh* (fig. 11.) be the representation given of a parallelogram which is similar to *DFGH*; it is required to determine the Vanishing Line of the plane of the original of *dfgh*, its Center, and Direct Distance.

Produce *df* and *hg* to* meet in *L*; and *dh*, *fg* to meet in *N*; and join *LN*; then (art. 27.) *L* is the Vanishing Point of the originals of *df* and *hg*, and *N* is the Vanishing Point of the originals of *dh* and *fg*; *L* and *N* are therefore (art. 37.) points in the Vanishing Line of the Original Plane, and *LN* is that line.

Again, join *FH*, and *fh*; *fdh* is the representation of a triangle similar to *FDH*, and the Center of the Vanishing Line, and its distance from the Point of View, may be found by art. 54.

Or, join *DG* and *dg*; upon *NL* describe (E. 33. 3.) a segment of a circle *LON*, capable of containing an angle equal to the angle *FDH*: At the point *N*, make the angle *LNP** equal (E. 23. 1.) to the angle *FDG*; and join *PL*; produce *dg* to meet *LN* in *M*; join *P,M*, and produce *PM* to meet the circumference in *O*; draw *OK* perpendicular to *LN*: *K* is the Center of *LN*, and *KO* is equal to its Direct Distance.

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* If *df* and *hg* do not meet, when produced, they are parallel to the Vanishing Line. If, therefore, from *N*, *NL* be drawn parallel to *hg* or *df*, it will then be the Vanishing Line.

For join *OL* and *ON*. Then, since the angle *LOP* is equal (E. 21. 3.) to the angle *LNP*, which was made equal to the angle *FDG*, therefore the angle *LOP* is equal to the angle *FDG*: And the angle *LON* is, by the construction, equal to the angle *FDH*: But (art. 44.) *LM* and *LN* must subtend, at the Point of View, angles equal to the angles *FDG*, *FDH* respectively. The Point of View is, therefore, at the same perpendicular distance from *LN*, that the point *O* is.

62. *Cor.* If the representation *dfgh* of a parallelogram, the Center *K*, and the Direct Distance of its Vanishing Line, were given, the ratio of two adjacent sides of the original figure might thus be determined.

1st. If *df* and *hg* be parallel to each other, produce *fg* and *dh* to meet in *N*, join *N,K* and *d,g*; and let *dg* produced, meet *NK* produced in *M*: Draw, from *K*, *KO* perpendicular to *MN*, and make it equal to the Direct Distance: join *ON*; then is the original of *df* to that of *fg*, as *MN* is to *NO*.

2ndly. Let *df* and *hg* meet, when produced in *L*;

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* If the original parallelogram be rectangular, the point *P* is more readily determined, by drawing *LI* (fig. 11. No. 2.) touching (E. 17. 3.) the circle in *L*, making *LI* (E. 12. 6.) a fourth proportional to *DF*, *FG* and *NL*; and joining *IN*; *IN* will then cut the circumference in the point *P*.

join *N*,*L*, and *K* will necessarily lie in *NL*; draw *KO* as before; join *O*,*L* and *ON*; about the triangle *LON* describe (E. 5. 4.) the circle *LONP*; join *d*,*g*, and let *dg* produced meet *LN* in *M*; join *OM*, and produce it to meet the circumference in *P*; lastly, join *PN* and *PL*: Then is the original of *df* to that of *fg*, as *Nl* is to *PL*.

PROP. VI.

63. *Problem.* The representation being given of a trapezium, which is similar to a given trapezium, to determine the Vanishing Line of its plane, the Center of that line, and its Direct Distance.

Let *dfgh* (fig. 10.) be the representation given of a trapezium, which is similar to *DFGH*; it is required to determine the Vanishing Line of the plane of the original of *dfgh*, its Center, and its Direct Distance.

Join *D*,*G* and *F*,*H*, and let *DG*, *FH* meet in *X*; also join *dg*, and *fh*, and let *dg*, *fh* meet in *x*; then since the original trapezium is similar to *DFGH*, the original of *dx*, is to that of *xg*, as *DX* is to *XG*, i. e. in a given ratio; and the original of *hx*, is to that of *xf*, as *HX* is to *XF*, i. e. in a given ratio: Find therefore (art. 54.) the Vanishing Points *M* and *L*, of the originals of *dg* and *fh*, and join *LM*; *LM* is the Vanishing Line (art. 37.) of the plane in which lie the originals of *dg* and *hf*, i. e. of the plane of the original trapezium.

Again, the original of the triangle *dxh*, is similar to the triangle *DXH*; find therefore the Center *K*, and the Direct Distance *KO*, by art. 54.

64. *Remark.* The articles of the preceding section are of very extensive use in examining pic-

tures,* supposed to be drawn according to the rules of Perspective. The object of such examination may be either to copy a painting, to judge fairly of its effect, to deduce the relative magnitude of the parts represented, or to compare the copy with its original. In each of these cases, it is necessary to know the Center, and the Principal Distance, of the Picture, and the position of the Vanishing Line of the ground plane, which the preceding propositions often enable us to determine. The second case must frequently occur. For many paintings appear to great disadvantage when seen from a point, which is not their proper Point of View, and great injustice is done to the artist by judging of them from such a situation. The Perspective Plane is commonly made perpendicular to the horizon, and the following maxims, deduced from what has been demonstrated, apply to pictures drawn upon that supposition.

1. The representations of vertical straight lines are perpendicular to the Base and Vanishing Lines of the ground plane (art. 16.); the *position* of the latter may therefore be found from the former.

2. The representations of straight lines which are parallel to each other, and situated on the ground plane, meet in the Vanishing Line of that plane, (art. 27. and 37.) which may therefore be found whenever the picture exhibits two pairs of such parallels.

3. If one side of the representation of a rectangle, which is either on the ground plane, or parallel to it,

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* Neither the Greeks nor the Romans are known to have been in possession of the mathematical principles of Perspective. It would not be an uninteresting task to examine the remains of ancient Painting, as far as it could be done by the rules of this section.

be parallel to the Base Line, or if it meet the representation of any vertical straight line at right angles, the other side, produced, passes through the Center of the Picture, (art. 28.) which may often thence be determined.

4. And if that rectangle be a square, its diagonals, when produced, meet the Vanishing Line of the ground plane, in two points, each at a distance from its Center equal to the Principal Distance (art. 55.)

5. If the representation be given of a rectangle on the ground plane, no side of which is parallel to the base of that plane, its Vanishing Line may be found by joining the two points in which the representations of the opposite sides meet: And if on the straight line joining those two points as a diameter, a semicircle be described, the Principal Distance is equal to the perpendicular distance of some point in the curve from the diameter.

6. If therefore the representation of another such rectangle be given, and a semicircle be described in like manner, a straight line drawn from the intersection of the two curves perpendicular to the Vanishing Line, will give the Center and Principal Distance of the Picture.

7. Hence, if the representation of a square situated on the ground plane be given, the Vanishing Line of that plane, together with its Center and Distance, may be found by this method: For the diameters of a square bisect each other at right angles, and their parts may be considered as the sides of another rectangle.

8. If the representation be given of a rectangle on the ground plane, the ratio of the sides of which

is known, the same things may determined by art. 61.

9. If a divided straight line represent a line on the ground plane, the ratio of the originals of its parts may be determined, when the Vanishing Line of that plane is given (art. 53.) For the point in which the given Vanishing Line is met by the given representation, is the Vanishing Point of the original line.

10. If the representation of any vertical straight line be given, and also that of a divided straight line on the ground plane, the ratio of the parts of which is known, the Vanishing Line of the ground plane may be found (art. 54.)

11. If the Base and Vanishing Lines of the ground plane be given, the ratio of any two finite vertical straight lines may be ascertained from their representations, by means of art. 49.

12. If the ratios be known of three straight lines each to each, situated in the same plane, and their representations and three Vanishing Points be given, the Center and the Direct Distance of the Vanishing Line of their plane may be found.

For the three Vanishing Points are necessarily in the same straight line, which is the Vanishing Line of their plane. And because the ratios of the original lines are given, the ratios of the distances of the Point of View from the several Vanishing Points (art. 26.) are also given. But if the distance between any two of the Vanishing Lines be considered as a base, the locus of the vertex of a triangle described upon it, having its sides in the given ratio to each other of the respective distances of the Point of View from those two Vanishing Points, is the circumference of a determinate circle (Simpson's Alg. p. 334.): And the Point of View is at the same distance from the Vanishing Line as some point in that circumference.

Again, if the distance between two other Vanishing Points be taken as a base, the Point of View may be shewn to be at the same distance from the Vanishing Line, as some point in the circumference of another circle determined in the same manner as the former. A straight line, therefore, drawn from the intersection of these two circles perpendicularly to the Vanishing Line, will meet that line in its Center, and be equal to its Direct Distance. But as this case, although elegant in its construction, can seldom occur in practice, the investigation of it is not given at length. For the same reason, another of the inverse problems of Perspective, admitting of a construction equally elegant, has been omitted. It is that in which the representation is given of a rectangular parallelepiped, not having any one of its sides perpendicular to the Perspective Plane, to find the Center of the Picture, and the Principal Distance, together with the species of the original object. The solution of this problem may be deduced from what has been already demonstrated.

SECTION III.

On the Appearance of Pictures, when seen from a Point which is not their proper Point of View.

PROP. I.

65. *Theorem.* A given indefinite straight line will have the same representation, whatever point in a straight line parallel to it be taken for the Point of View.

For the representation (art. 14.) is the intersection of the Perspective Plane, with a plane passing through the Point of View and the original line; and this plane will manifestly remain the same, whatever point in the straight line parallel to the original line be taken for the Point of View.

PROP. II.

66. *Theorem.* If an original finite straight line be parallel to the Perspective Plane, its representation will be of the same length, whatever point in a plane, parallel to the Perspective Plane, be taken for the Point of View.

For the Principal Distance and the perpendicular distance of the Point of View from the plane containing the given original line, will in this case remain the same; and therefore (art. 19.) the representation will also remain the same.

66. *Remark.* Hence, if the side of a building,

which is parallel to the Perspective Plane, be delineated, the justness of the representation will not be affected by the circumstance of the drawing being placed on the wall of a room, so that a straight line drawn perpendicular to the wall from the eye, shall not meet it in the point which was the Center of the Picture.

PROP. III.

68. *Theorem.* If a straight line in the Perspective Plane, seen from points at different perpendicular distances from it, be always considered as the representation of some original straight line in any the same given plane, which is parallel to the Perspective Plane, the difference in length between the representation and its original will be the greater, the less is the Principal Distance.

Let *ab* (fig. 2.) in the Perspective Plane *PL*, when seen from different points *O*, continue to be the representation of some straight line $\alpha\beta$ in *pl*, which is parallel to *PL*: The excess of $\alpha\beta$ above *ab* will be greater in proportion as *Od*, the Principal Distance, is less.

For it may be shewn, as in art. 19, that

$$\alpha\beta : ab :: O\delta : Od$$

∴ The excess of $\alpha\beta$ above $ab : ab :: d\delta : Od$ (E. 17. 5.)

But if *Od* be diminished, the ratio of $d\delta$ to *Od* (E. 8. 5.) is increased, since $d\delta$ remains the same; and therefore the ratio of the excess of $\alpha\beta$ above *ab* to *ab* is also increased; and since *ab* remains the same, the excess of $\alpha\beta$ above *ab* must (E. 10. 5.) be increased.

69. *Remark.* If the side of a building parallel to the Perspective Plane constitute the whole of the

representation, the Picture will have the same appearance, at whatever perpendicular distance from it the Point of View be taken. For it is manifest, from art. 21, that the proportion of the several parts represented will remain the same, the only difference being that of apparent magnitude. The case is the same, when the Picture represents several objects all at equal distances from it or nearly so.

PROP. IV.

70. *Problem.* To determine the Direct Distances, at which, if the given representation of a given angle, situated in a given plane, be seen, it shall continue to represent an angle of the same magnitude in the same given plane.

Let *LN* (fig. 11.) be the Vanishing Line of the given plane which contains the given angle, and let *LdN* be the given representation: It is required to determine the several direct distances of *LN*, from which if *LdN* be seen, it shall continue to represent an angle of the same magnitude, in the same Original Plane.

Upon *LN* (E. 33. 3.) describe a segment of a circle *LON*, capable of containing the given original angle: Take any point *O* in the arch *LON*, and draw (E. 12. 1.) from *O*, *OK* perpendicular to *LN*. If then (art. 44, and E. 21. 3.) a straight line be drawn from *K* perpendicular to *LN*, in the plane which determines *LN*, and be made equal (E. 3. 1.) to *KO*, it will be one of the Direct Distances, and its extremity one of the Points of View required.

PROP. V.

71. *Theorem.* If the representation of a rectangle,

be seen from any point either in the Direct Distance of the Vanishing Line of the original figure, or in that line produced, it is still the representation of a parallelogram situated in the same plane, but of which the two adjacent sides have not always the same ratio to each other, as the adjacent sides of the original rectangle.

Let (fig. 12 and 13.) *dfgh* be the representation of a rectangle, of which *kKϰ* is the Vanishing Line and *K* its Center; *dfgh* when seen from any point in the Direct Distance of *kKϰ*, or in that line produced, will represent a parallelogram, but the originals of *df* and *fg* will not always have the same ratio to each other as the adjacent sides of the rectangular parallelogram, first represented by *dfgh*.

First, let *df* (fig. 12.) and consequently *gh*, be parallel to *kKϰ;* therefore (art. 17.) the originals of *df* and *gh* are parallel to the Base Line of the original plane, and since *fdh* represents a right angle, *dh* produced (art. 38.) passes through *K*: But *K* continues to be the Center of *kKϰ*, whatever point in the Direct Distance, or in that line produced, be taken for the Point of View, and therefore *dfgh* continues in this case to represent a *rectangular* parallelogram.

Again, from *K* draw (E. 11. 1.) *KO* perpendicular to *kKϰ*, and make it equal (E. 3. 1.) to the Direct Distance of the Vanishing Line of the original figure, and in *KO*, or *KO* produced, take any point *T;* join *fh* and produce it to meet *kKϰ* in *M;* also join *OM*, and *TM:* Then (art. 62.) the original of *hd* is to that of *df*, as *OK* to *KM*, when the Point of View is at the distance *KO* from *K;* but when it is at the distance *KT*, the original of *hd* is to that of *df*, as *TK* to *KM*: And the ratio of *OK* to *KM*, is greater or less than the ratio of *TK* to *KM*, according as *OK* (E. 8. 5.) is greater or less than *TK*.

But if *df* (fig. 13.) be not parallel to *Kk*, produce *df* and *dh* to meet *Kk* in *L* and *N* respectively; then since, by the supposition, the Vanishing Line remains the same, *dfgh* will still (art. 27.) represent a parallelogram, whatever point in the Direct Distance be the Point of View. Draw *KO*, as before, and make it equal to the Direct Distance of the Vanishing Line of the original figure, and in *KO*, or *KO* produced, take any point *T*; join *LO*, *LT* and *NO*, *NT*; then is the angle *LON* (art. 44.) a right angle, and the angle *LTN* is equal to the angle represented by *fdh*, when seen from a point at the distance *KT* from *K*, and which (E. 21. 1.) is greater or less than a right angle, according as *KT* is less or greater than *KO*.

Again, the lines represented by *df*, seen from Points of View at the distances *OK* and *TK*, are to each other (art. 26.) as *OL* to *TL*: and the lines represented by *dh*, seen from the same Points of View, are to each other as *ON* to *TN*. If it be possible, let the ratio* of the two original lines seen from the first of these points be necessarily the same as their ratio when seen from the second: Then (E. 16. and 11. 5.) *OL* has constantly to *ON* the same ratio as *TL* has to *TN*, which is absurd: For the vertexes of triangles described upon the same base and having their sides in a constant ratio will always be found in the arch of a circle, (*Simpson's Geom. p. 220. or Alg. p. 334.*) and not in a straight line.

---

* The ratios of the original lines may be exhibited as in fig. 13. No. 2. determined from art. 62.

PROP. VI.

72. *Theorem.* If the representation of a rectangle be seen from any point in a straight line, passing through the first Point of View, parallel to the Vanishing Line of the original figure, it is still the representation of a parallelogram situated in the same plane; but two adjacent sides of that parallelogram have not always the same ratio to each other, as the adjacent sides of the original rectangle.

Let $dfgh$ (fig. 12. and 13.) represent a rectangle, when $kK\kappa$ is the Vanishing Line and K its Center; if $dfgh$ be seen from any point in a straight line parallel to $k\kappa$ and passing through the first Point of View, it will still represent a parallelogram situated in the same plane as the rectangle first represented, but of which the adjacent sides have not always the same ratio to each other as those of that rectangle.

If df or dh be parallel to $K\kappa$, the proposition may be shewn to be true, in the same manner as the first case of the preceding theorem was demonstrated, and in the same manner as its second case was demonstrated, if neither df nor dh be parallel to $k\kappa$.

73. *Cor.* When fd (fig. 12.) is parallel to $k\kappa$, and fdK represents a right angle, if the Center of the Vanishing Line be κ, fdh will represent an acute angle, but if k be that Center, fdh will represent an obtuse angle.

For if $d\kappa$ and dk be joined, $fd\kappa$ in the former case, and fdk in the latter, represent each a right angle;

and *fdK* is less than *fdx*, but greater than *fdk*. Thus *kd* will appear to direct itself toward the eye, as the eye moves along the line described in the proposition; and hence, in a portrait, if the axes of the eyes be drawn as perpendicular to the Perspective Plane, they will appear to look at the spectator as he moves in a line parallel to the wall on which the portrait is fixt.

74. *Remark.* The two preceding propositions serve to explain some of the appearances of pictures, seen from points which are not their proper Points of View.

Let *ABCca* and *GEFfg* (fig. 14.) be the delineations of two houses on a Perspective Plane perpendicular to the horizon, *K* being the Center of the Picture. The house represented by *ABCca*, has one of the sides parallel, and the other perpendicular to the Perspective Plane; and both the sides of the other house are oblique to the Perspective Plane, but at right angles to each other.

If the eye move in a straight line drawn from *K* perpendicular to the plane of the picture, *ABC* will still represent a right angle, but the relation of the sides containing that angle will change. This variation is not very easily perceived, but as in the representation of the other building, both the angle represented by *GEF*, and also the sides containing it vary, the alteration in the appearance is more readily noticed.

If the eye move in a straight line parallel to *Dd*, at the same height above the ground, and such that the Principal Distance is always the same as it was originally, as the Center moves from *K* towards *D*, the angle represented by *ABC* becomes obtuse, and that represented by *GEF* becomes acute. As the Center moves from *K* toward *d*, the former angle becomes acute; the latter becomes first obtuse, then, at one point, a right angle, (see fig. 12 and 13.) and lastly acute. The relation of the sides containing each of the angles, varies in both these cases.

If the picture be hung up at too great an height, what was the Vanishing Line of the horizon, becomes the Vanishing Line of an inclined plane, and what was intended for the ground will appear an acclivity. This circumstance often gives an absurd appearance to portraits, where the principal figure is represented as in a room, the floor, or carpet, of which is also a part of the picture. The person drawn seems, in this case, to be in danger of slipping down the inclined plane presented to the eye.

If, on the contrary, the picture be hung up below the usual height of the eye, what was intended for the ground plane will seem to be a declivity, and to slope from the eye like the side of a mountain seen from its top.

When the plane of the picture is too high, *bBC* and *bcC* (fig. 14.) will represent acute angles, and *Bbc* and *cCB* obtuse angles: And when the place of the picture is too low, *bBC* and *bcC* will represent obtuse angles, and *Bbc* and *cCB* acute angles.

It is most necessary for the eye to be situated in the true Point of View, when the Picture is drawn upon an inclined plane, or upon a curved surface. In these cases, lines which are meant to be the copies of objects which we know to be necessarily perpendicular to the horizon, will not appear to represent such originals, unless they be seen from the point assumed by the painter.

It is evident from art. 70 that the picture of a rectangular parallelepiped, which has its base perpendicular to the Perspective Plane, as that in fig. 8, No. 2, will constantly represent a rectangular parallelepiped, to an eye placed in any point of the circumference of a determinable circle: And Doctor Taylor has pointed out one case, in which the consideration of this property might be of use. It is that of the scenes of a theatre, where it is intended that the picture of a house, or of any other solid, of the form of a rectangular parallelepiped, should represent the same kind of object to the eye of every one of the spectators; the plan of that part of the building in which they are placed, having, as is proper and usual, a circular boundary.

SECTION IV.

On the Delineation of Shadows.

PROP. I.

75. *Problem.* The representation of a given finite straight line being given, to find the representation of its shadow,* cast upon a given Original Plane; the intercepted rays diverging from a given luminous point.

First, let the original of *AB*, (fig. 15.) the shadow of which is required to be represented, be parallel to the given plane on which the shadow is cast: Find (art. 30.) the representation, *L*, of the given luminous point, and *M,a,b* those of the points in which the given plane is met by straight lines drawn perpendicular to it from the originals of *L*, *A* and *B*. Join *LA* and *Ma*, and let *LA* and *Ma* meet, when produced, in *S*; also join *LB* and *Mb*, and let *LB* and *Mb*, when produced, meet in *H*; lastly, join *SH*; *SH* is the representation of the shadow required.

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* Neither in this nor the following proposition, is the original line supposed to pass through the source of the rays, in which case its shadow is a point. Accurately speaking, neither a point nor a line can be said to cast a shadow. That which is determined in this section, is properly the boundary of the shadow of a surface. It might also be objected, by those who are disposed to cavil, or by such as affect extreme precision, that neither a point nor a line, mathematically defined, can have any Perspective Representation. In order to preclude any such objection, we might have begun this treatise, by determining the representation of a surface, contained by more than one line. What we have called the representation of a line would then have been the *boundary* of the representation of the surface, and what has been termed the representation of a point, would have been the extremity of that boundary. There is, however, a conveniency in the more generally received mode of considering the subject, and there does not seem to be any sufficient reason for departing from it.

For, by the postulate, *LA* represents the ray of light intercepted by the original of the point *A*; and the originals of *LM* and *Aa* are (E. 6. 11.) parallel, and, therefore, in the same plane with each other, and with the originals of *LA* and *Ma*, which lines, therefore, meet in the point of the given plane, which is the shadow of the original of *A*; therefore *S*, the point in which their representations meet, is the representation of the shadow of the original of *A*. In the same manner *H* may be shewn to represent the shadow of the other extremity of the given line: It is, therefore, manifest, from the postulate, and from E. 3. 11. that *SH* is the representation of the shadow required.

But if the given original line be not parallel to the given plane, let *CD* be its representation, the line meeting the plane, in the original of *D*: Find the representation *N* of the shadow of the original of the point *C*, as in the former case, and join *DN*; *DN* may then be shewn to represent the shadow of the original of *CD*, by reasoning as before.

76. *Cor.* By the help of this proposition the representation may be found of the shadow of any given rectilineal figure, cast upon a given plane surface, when the rays intercepted diverge from a

given point; and if any of the lines bounding the plane figure be a curve, any number of points in the shadow of that line may be thus represented.

PROP. II.

77. *Problem.* To find the representation of the shadow of the original of a given point in the picture, cast upon a given plane, which is perpendicular to the Perspective Plane, when the rays of light are parallel to each other, and to a given straight line.

Let *A* (fig. 16.) be the given point in the picture, and *Mm* the Vanishing Line of the given perpendicular plane which receives the shadow, found by art. 36; it is required to find the representation of the shadow of the original of *A*, cast upon the plane, of which *Mm* is the Vanishing Line, the rays of light being supposed parallel to each other, and to a given straight line.

Find (art. 30.) the representation, *a*, of the point in which the given plane is met by a straight line drawn perpendicular to it from the original of *A*.

First, let the given straight line to which the rays are parallel, be itself parallel to the Perspective Plane, and let *RY* be its representation found by art. 50; through *A*, draw (E. 31. 1.) *AP* parallel to *RY*, and through *a* draw *aP* parallel to *Mm*, and meeting *AP* in *P*; *P* is the representation of the shadow of the original of *A*.

For it is manifest that *AP* represents the original of the ray passing through *A*, and that *aP* represents the shadow of the original of the line *Aa*; therefore *P* represents that of the original of *A*.

But if the given straight line, to which the rays of light are parallel, be not parallel to the Perspective Plane, its Vanishing Point *L* or *l* must first be found. Let *C* be the Center of the Picture, which must be in *Mm*, (art. 19.) the given plane being perpendicular to the Perspective Plane; through *C* draw (E. 11. 1.) *FCE* perpendicular to *Mm*, and meeting the Base Line *EB* in *E*. Make *CF* (E. 3. 1.) equal to the Principal Distance. From *F* (Dem. of art. 36.) draw the straight line *FM*, making with *Mm* the angle *FMm* equal to the inclination of a plane passing through *CE*, (which is perpendicular to the given plane, and also passes through any straight line drawn in the given direction of the rays) to the Perspective Plane; and let the source of the parallel rays and the Point of View be first supposed to lie on contrary sides of the Perspective Plane. From *Mm* cut off *MG* (E. 3. 1.) equal to *MF*, draw (E. 11. 1.) *ML* perpendicular to *MG*, and at the point *G* make the angle *LGM* (E. 23. 1.) equal to the inclination of the given straight line, in the direction of the rays, to the given plane of which *Mm* is the Vanishing Line; and let *GL* meet *ML* in *L*. *L* is the point required.

For it is evident, from art. 35, and the note to art. 33, that *ML* is the Vanishing Line of the plane, of which *CE* is the base, and which contains a straight line drawn in the given direction of the rays; and also, that *M* is the Center, and *MF* equal to the Direct Distance of that Vanishing Line. But *MG* was made equal to *MF*: and the angle *LGM* equal to the inclination of the given straight line to the given plane; therefore the angle *GLM* is equal to that which the same given straight line makes with the

Base Line *CE:* and *L* is the Vanishing Point (art. 40.) of that straight line, and (art. 27.) of all other straight lines parallel to it.

In the same manner the point *l* may be determined, which is the Vanishing Point of the parallel rays, when their source is on the same side of the Perspective Plane with the Point of View.

Let now *LA* and *Ma* be joined, and let *LA* and *Ma* produced meet in *S;* or if the Vanishing Point determined be at *l*, join *lA* and *ma*, and let *lA* and *ma* meet in *s;* then it may be shewn, as in the preceding problem, that *S* or *s* represents the shadow of the original of the point *A*.

78. *Cor.* By the help of this proposition, when the rays intercepted are parallel, and given in position, the shadow of any given straight line, or of any given rectilineal figure cast on a given plane perpendicular to the Perspective Plane, may be represented, as also any number of points in the shadow of a given curved line.

79. *Remark.* On account of the distance of the sun, its beams may be accounted parallel; and the above proposition is applied to determine the shadows of objects, intercepting the solar rays, when the Perspective Plane, as is most usual, is perpendicular to the ground plane on which the shadows are cast. If the shadow be received on a plane inclined to the ground plane, or on any surface whatever, the column of air darkened by the obstructing body must be first determined; and then the intersection of the representation of the inclined plane, or other surface, with that of the column, will give the representation of the shadow.

80. EXAMPLES.

1. In fig. 17, let the book represented by *aB* standing perpendicularly upon the table *DE*, be placed before the candle represented by *L*, and let it be required to draw its shadow.

M being the point in which the axis of the candle, supposed perpendicular to the horizon, meets the the table, join *Ma* and *LA*, and let them meet in *S*. In the same manner let *Mb* and *LB* meet in *H*: Then is *aSHb* the representation required.

2. Let it be required to determine the shadow of the gateway represented by *aBCc*, (fig. 18.) the light intercepted being that of the sun.

Let *MK* be the Vanishing Line of the ground plane on which the gateway stands, and *L* the perspective place of the sun's center. Draw *LM* perpendicular to *MK*; join *Ma* and *LA*, and let them meet in *S*; then *S* represents the shadow of the original of *A*, and in the same manner the other angular points of the picture of the shadow may be found. But the operation may be greatly abridged by considering, that if any original straight line be parallel to the plane on which its shadow is received, its shadow will be parallel to itself, and consequently the representation of the shadow will pass through the Vanishing Point of the original line.

3. Let the ladder represented by *PT* (fig. 18.) be placed against the wall represented by *NO*; let the Perspective place of the sun be *L*, as before; and let it be required to draw the shadow of the ladder, the steps being parallel to each other, and the limbs in which they are placed also parallel.

It is further supposed, that the wall, which supports the ladder, is perpendicular to the plane of the picture, the Center of which is K.

From any point D in PQ draw Dd perpendicular to MK, and let it meet Qd, drawn from Q parallel to MK, in d. Join Md, and let Md produced meet the wall No in p. Join Qp, pP; then is QpP the delineation of the shadow of the limb represented by PQ. Let pQ produced meet KM in Z; join ZT; and let ZT produced meet NO in r; join rR, then does TrR represent the shadow of the other limb. The shadow of any of the steps, as that represented by WI, may be drawn thus: Join LW, and let LW produced meet Pp in w; also join wK, and let wK meet Rr in i; then wi represents the shadow of the original of WI.

4. To determine the representation of the shadow of the pyramid, of which $MNOq$ (fig. 8. No. 2.) is the delineation, the sun being supposed to be in the plane of the picture, and the direction of its rays given.

Draw qS parallel to the given direction of the rays, and QS parallel to the Base Line AB, and let them meet in S. Then S represents the shadow of the vertex of the pyramid; and, if MS and NS be joined, MSN represents the shadow of the whole figure.

5. To delineate the shadow of the parallelepiped represented by Cb (fig. 19.) when that shadow is received upon the steps EK; the sun being supposed to be in the plane of the picture, and the direction of its rays given.

Produce ab to meet DE which represents the intersection of the lowest step with the ground; draw OP perpendicular to aO, and meeting the representation of the upper boundary of the original of the plane *DF* in *P*; draw *P2* parallel to *qO*, and so on; and produce the last perpendicular *VS*, until it meet, in *S*, *SL*, a straight line drawn through *B* parallel to the direction of the rays. Again, draw *cT* parallel to *aO*, and proceed as before. *TOPRVW* represents the shadow of the original of the solid *Cb*.

FINIS.

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ERRATA.

Page
7. l. 11. for *E. Def.* 1. 11. read *E. Def.* 3. 11.
16. l. 23. for *KD* read *Kd.*
19. l. 1. for *E.* 26. 8. read *E.* 26. 1.
25. l. 27. for *HC* read *HG.*
l. 33. dele & *GF : AB :: gf : ab* (*E.* 16. 5.)
26. l. 1. after *therefore* insert (*E.* 14. 5.)
31. l. 25. for *g* 30′ read *g* 30
l. 27. for *lk* read *l, k,*
33. l. 21. for *in* read *and*
38. l. 17. for *gL* read *gh*
39. l. 9. for *of ghl* read *of the angle ghl*
41. l. 10. between the latter *Dd* insert a comma
l. 13. for *Eg* read *EG*
45. l. 7. for *Nl* read *NL*
46. first line in the note, after *been* insert *fully*.

Fig. 1.

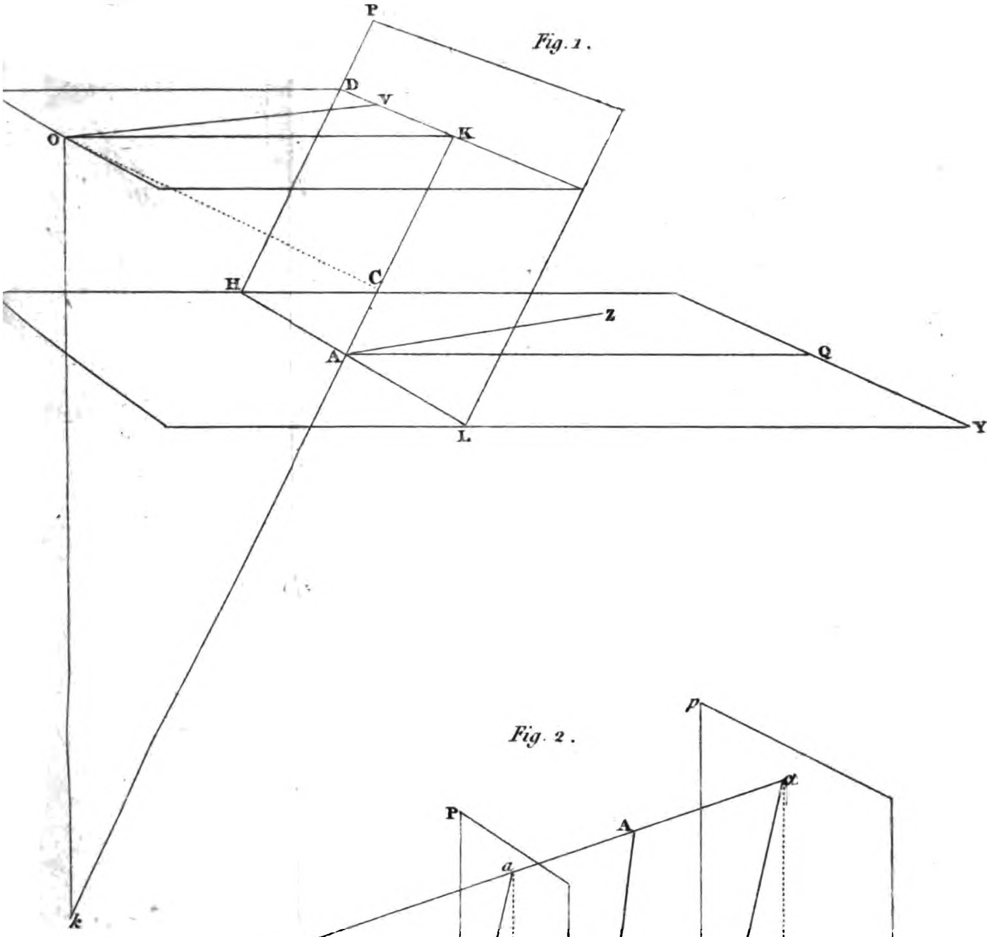

Fig. 2.

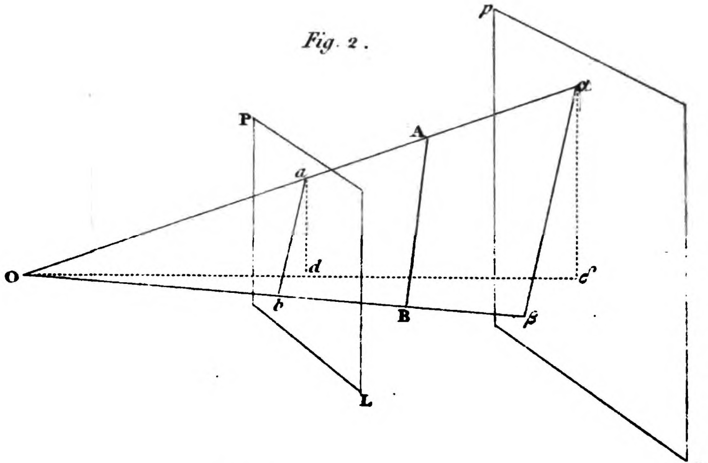

Fig. 3.

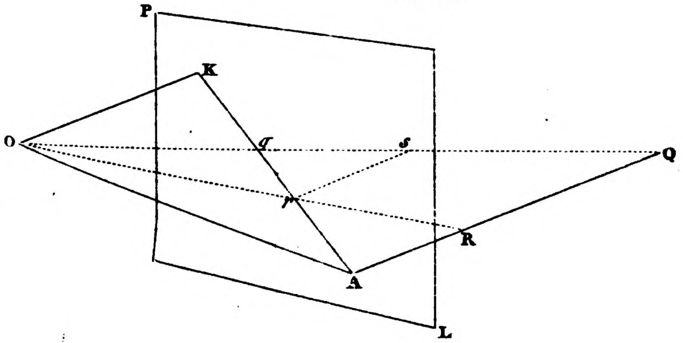

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Fig. 4.

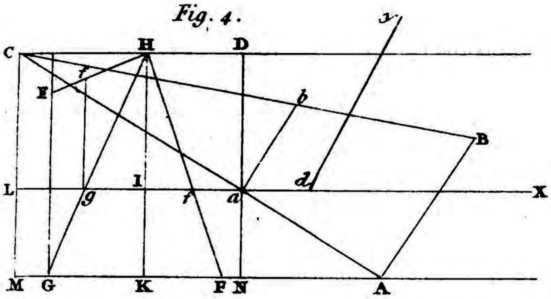
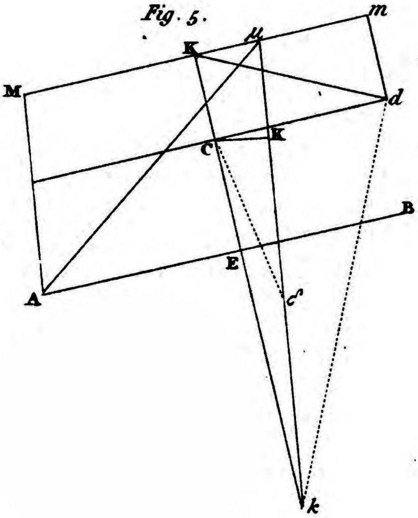


Fig. 5.



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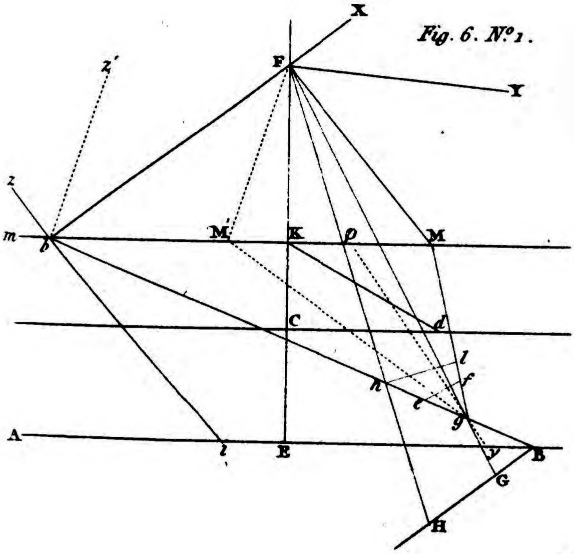

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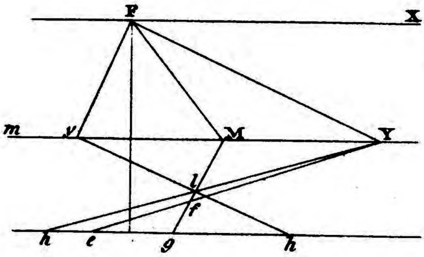

Fig. 7.

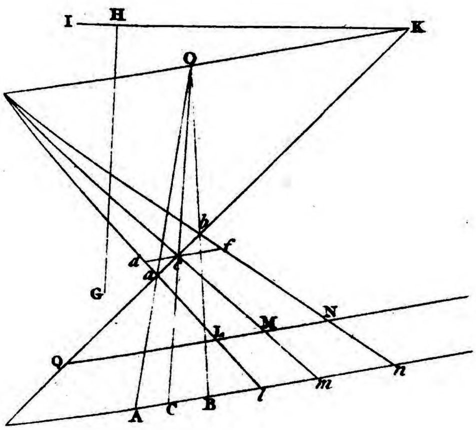

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Fig. 8. No. 1.

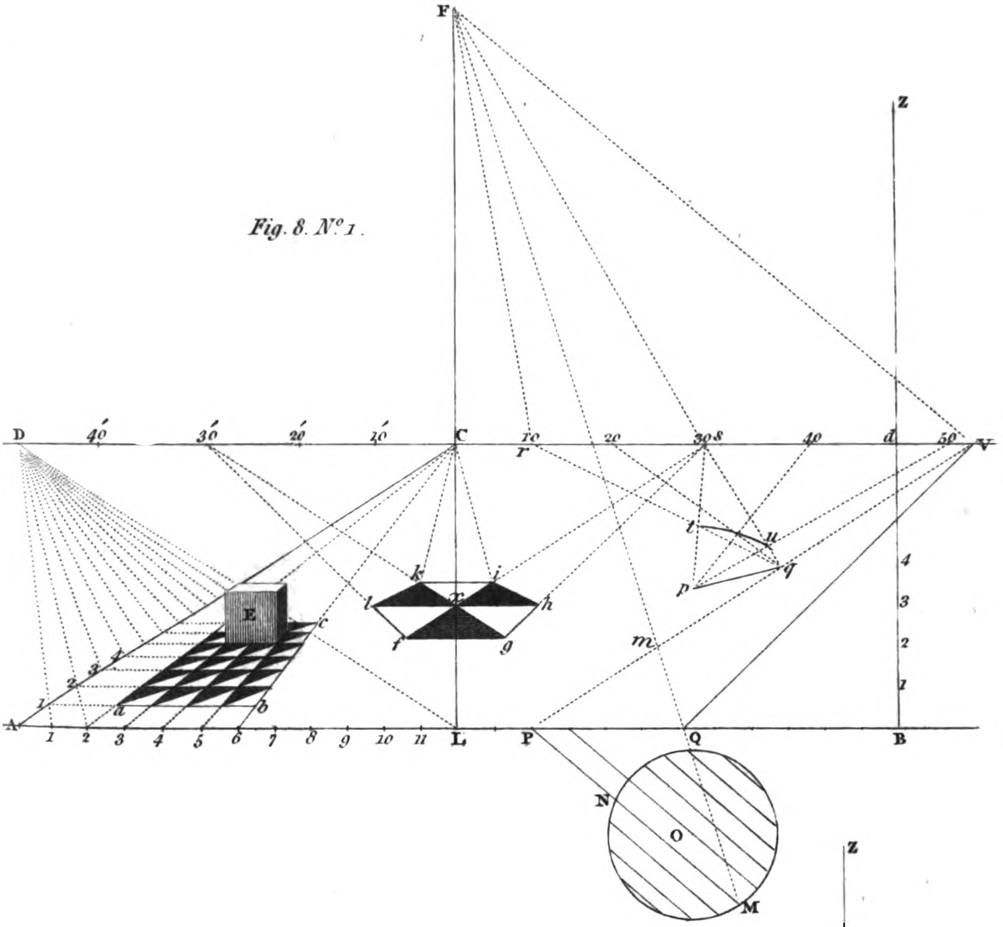

Fig. 8. No. 2.

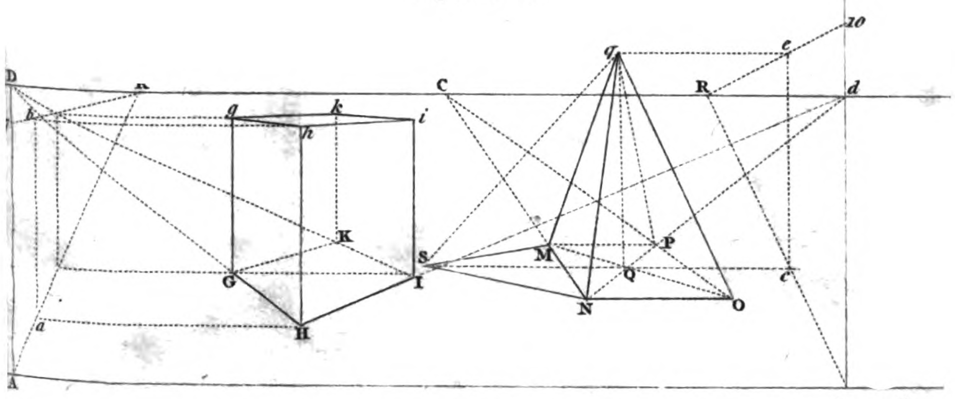

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Fig. 9.

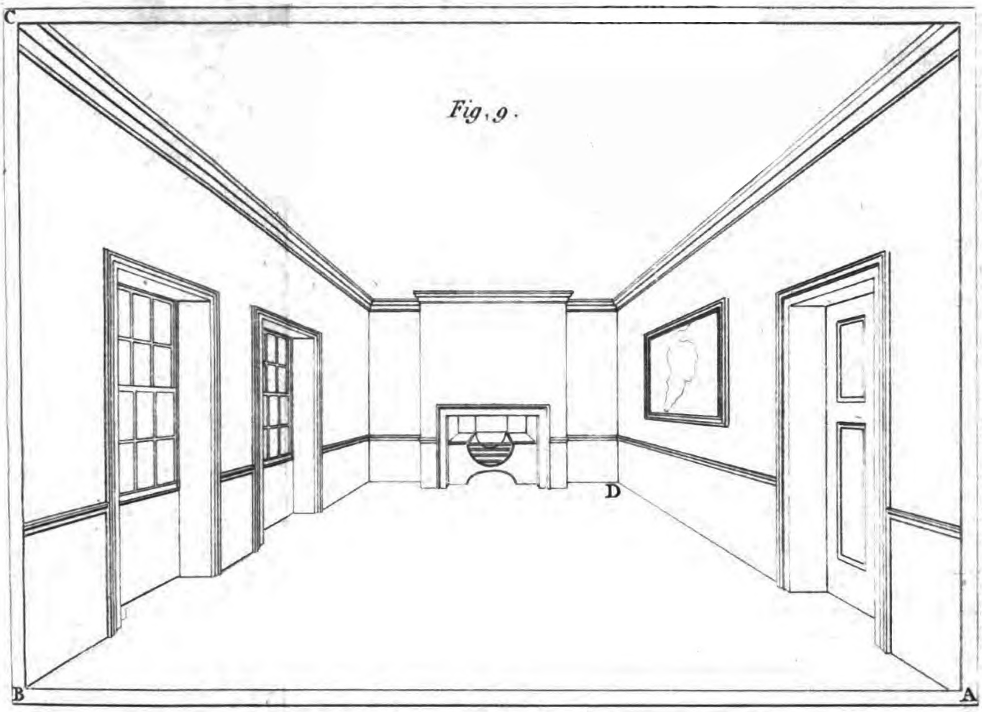

Fig. 10.

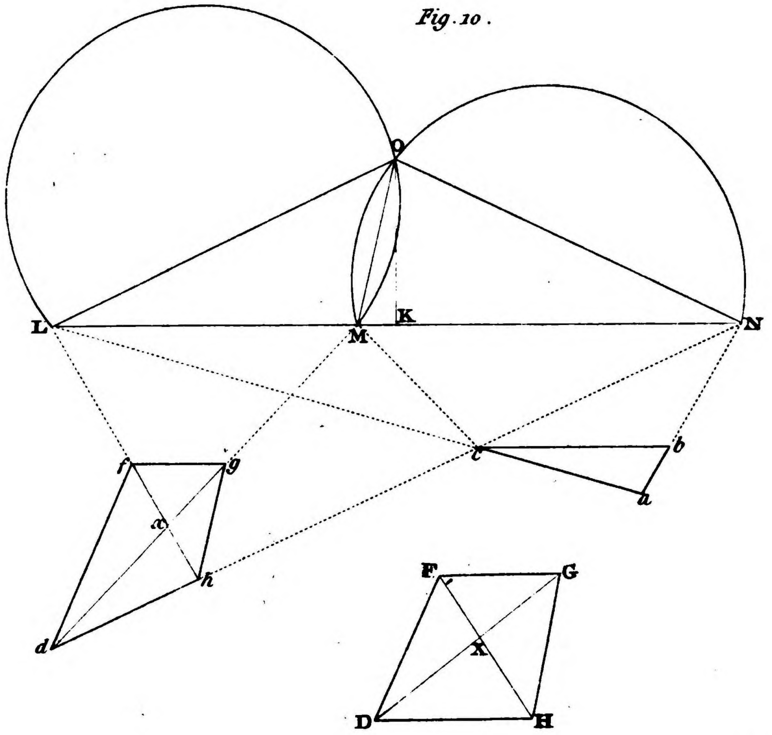

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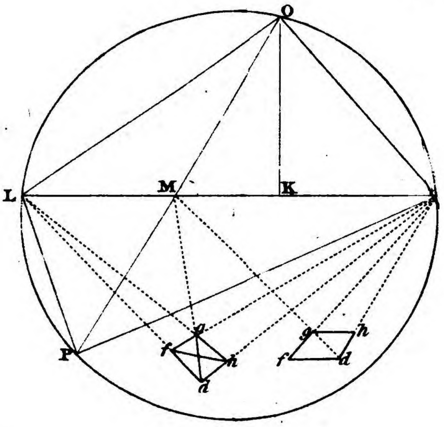
Fig. 11. N°. 1



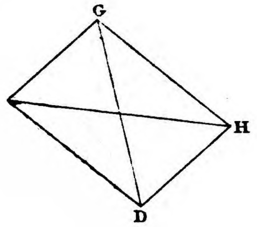
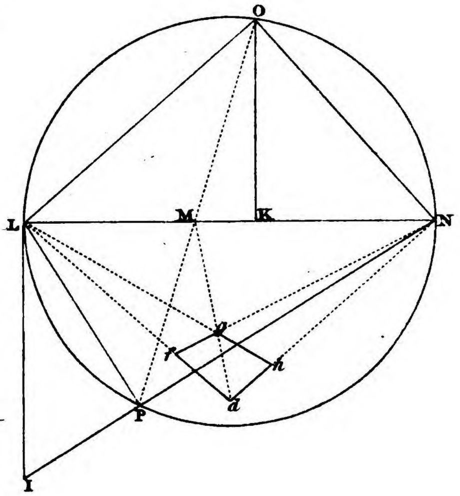


Fig. 11. N°. 2.



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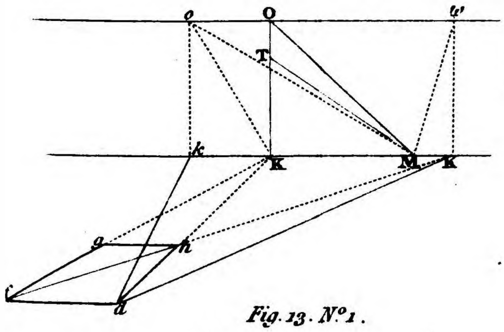

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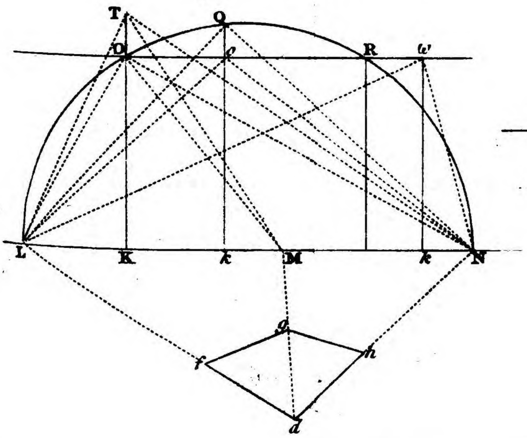

Fig. 13. N°. 2.

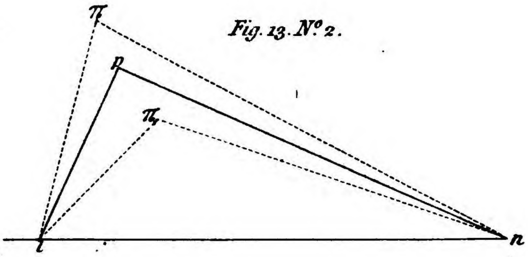

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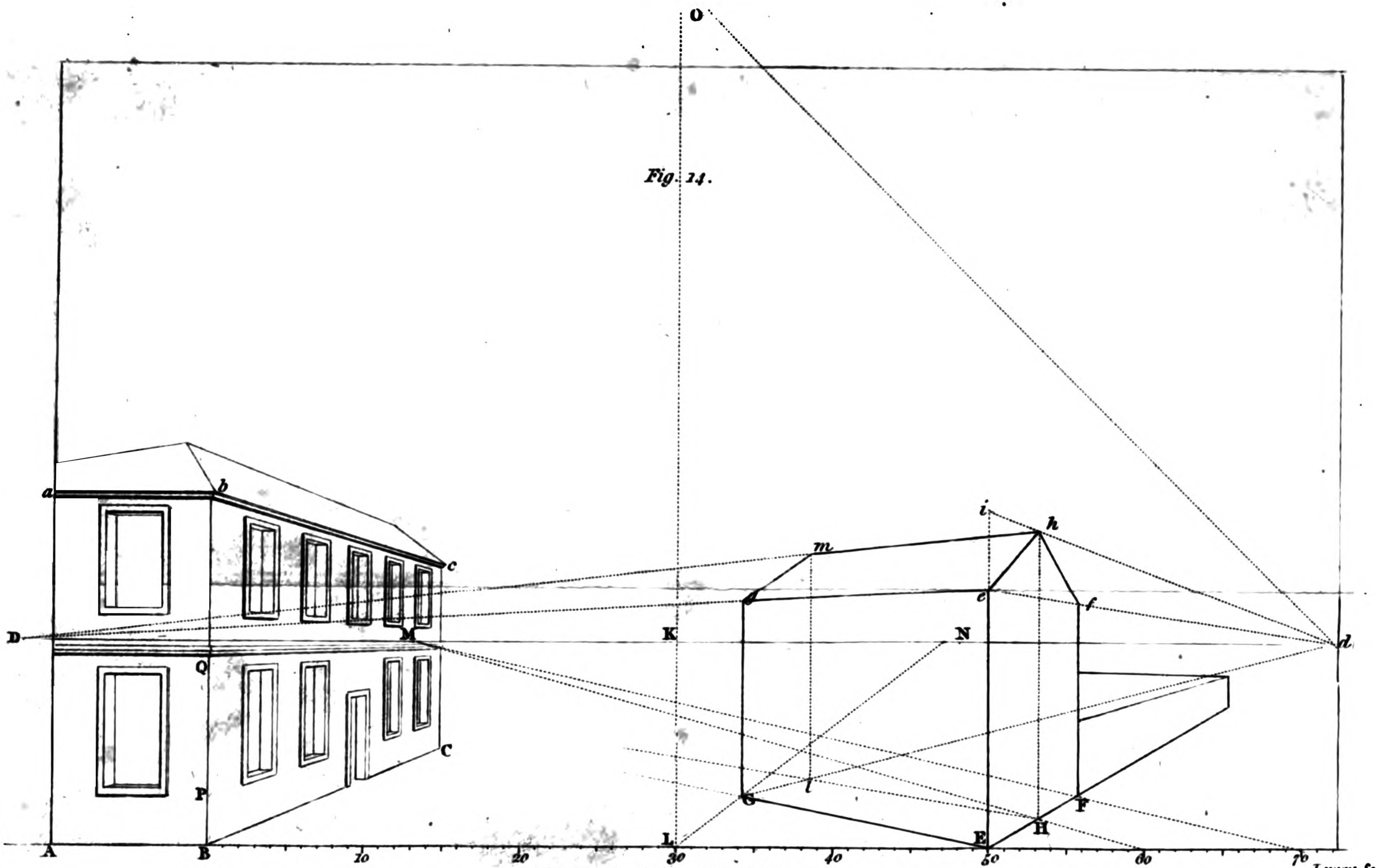
Fig. 14.



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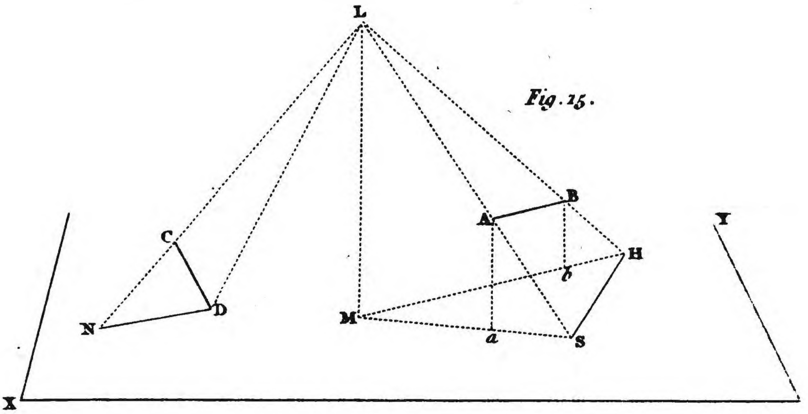
Fig. 15.

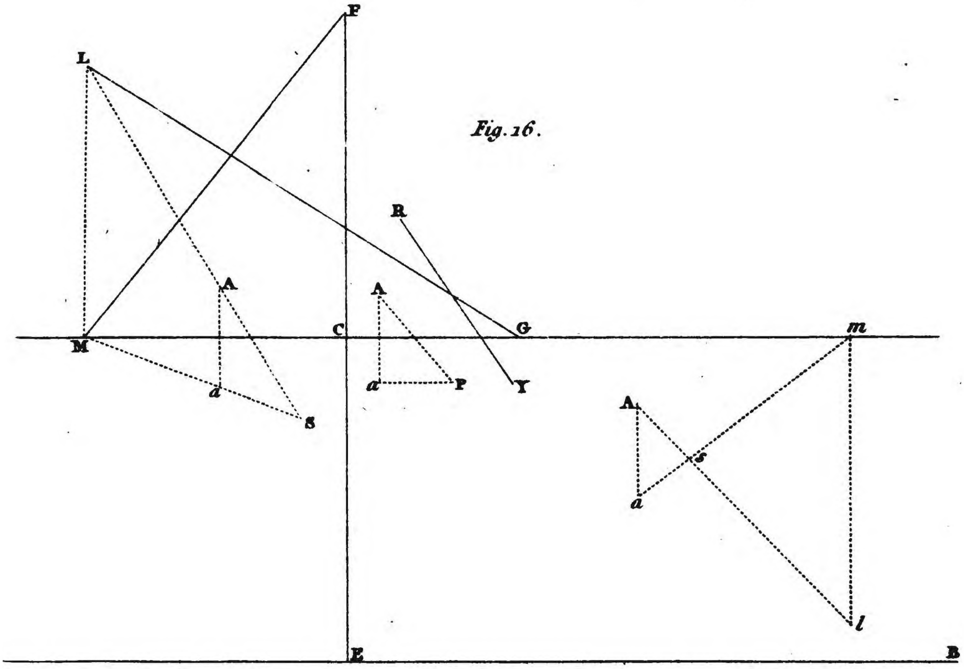
Fig. 16.

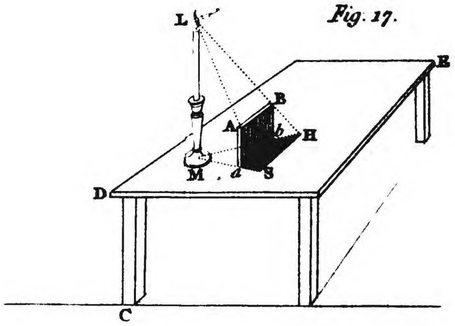
Fig. 17.

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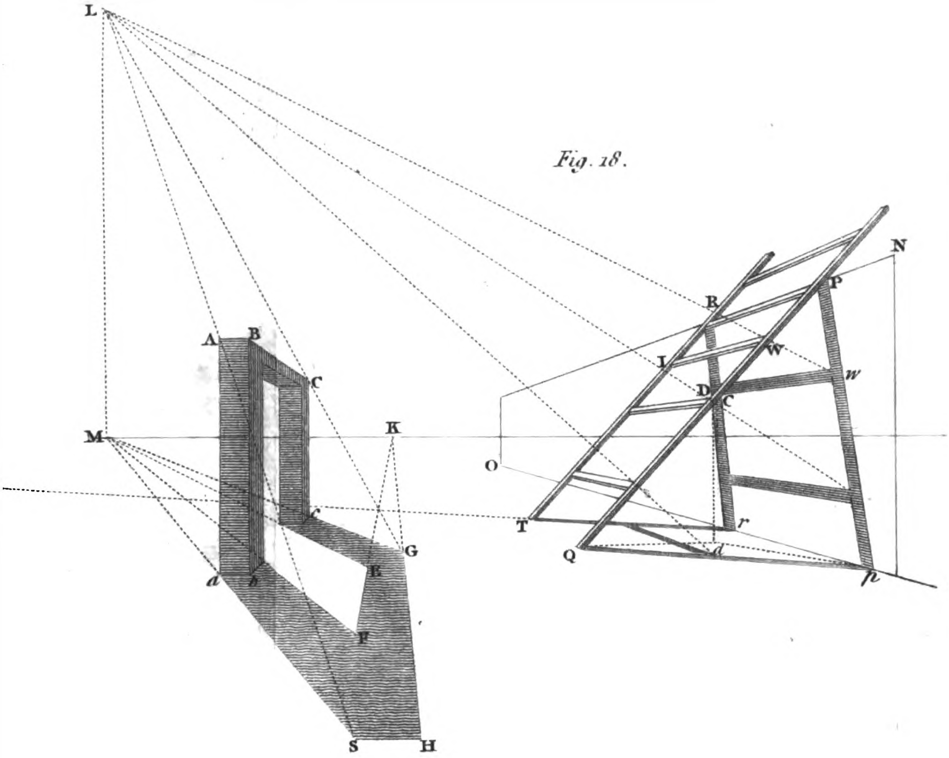

Fig. 18.

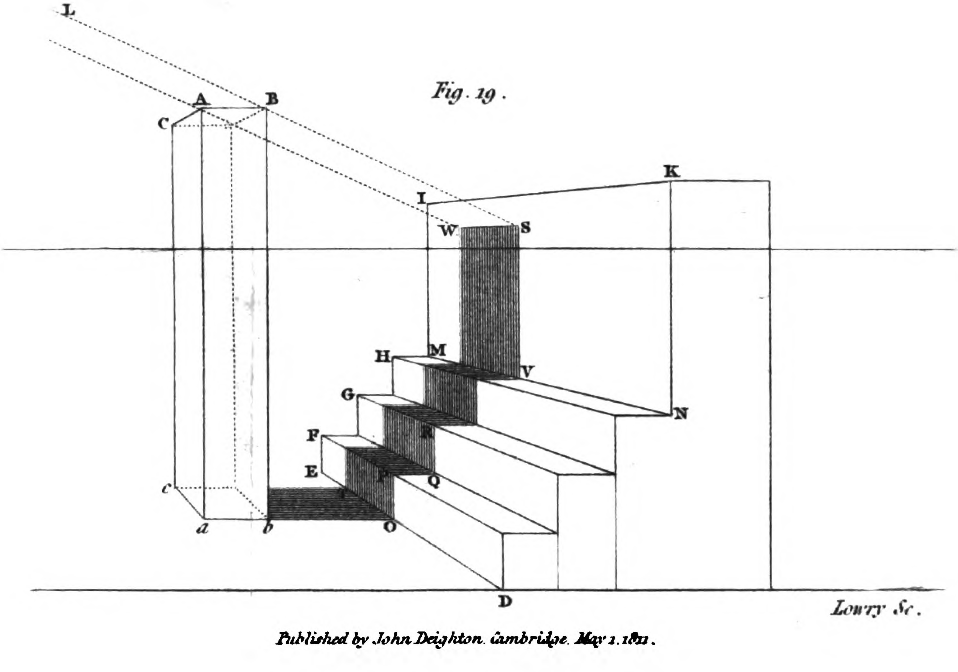

Fig. 19.

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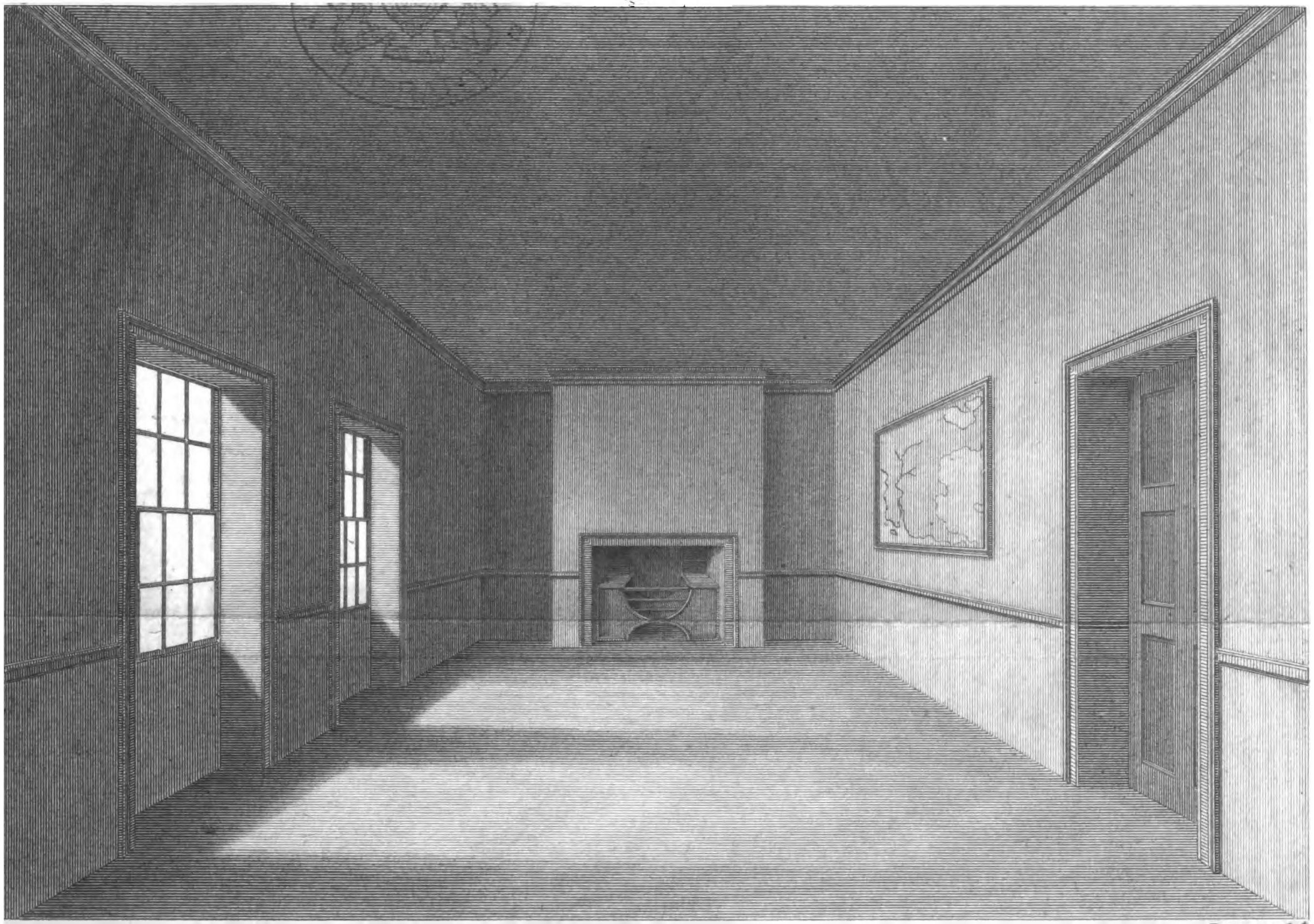
Fig. 20.



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The chamber delineated in Fig. 9, is here shaded to illustrate the remark p. 34.